PERCOLATION THEORY: THE COMPLEMENT OF THE INFINITE CLUSTER & THE ACCEPTANCE PROFILE OF THE INVASION PERCOLATION

A Dissertation
Presented to
The Academic Faculty

By

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics

Georgia Institute of Technology

August 2019

PERCOLATION THEORY: THE COMPLEMENT OF THE INFINITE CLUSTER & THE ACCEPTANCE PROFILE OF THE INVASION PERCOLATION

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A great quote to start the thesis

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ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor professor Michael Damron for his teaching, patience, motivation, enthusiasm, and immense knowledge. If I had not met him at indinana university, I would not have known the fun of probability and percolation theory. His teaching, advice and guidance have helped me in all time of Ph.D study and research. I could not have imagined my Ph.D study without him.

Besides my advisor, I would like to thank the rest of my thesis committee: professor Christian Houdré, professor Vladimir Koltchinskii, professor Mayya Zhilova and professor Robert Foley. Their questions and comments have given me great inspiration and help.

Finally, I appreciate to my wife, family and friends. If I had not met my wife Jiwon, I would not have started the Ph.D study and I would not finish Ph.D study without her support and help. Also, I thank my father, mother and younger sister for their endless love and support. Thanks to their support, I was able to continue my Ph.D. life and I was able to get through hard times.

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SUMMARY

We consider Bernoulli bond percolation (the detailed definition is in Chapter 1 and the introduction of Chapter 2). There have been studies on geometric features of the infinite (open) cluster and complement of an infinite (open) cluster in percolation theory on $\mathbb{Z}^d$. In Chapter 2, we try to determine properties of complement of an infinite cluster. To understand it, we introduce "shielded" vertices - those vertices whose incident edges are all closed, and "shielded" percolation - analysis of infinite connected components of shielded vertices. From the definition of shielded vertex, a shielded infinite component should be contained in the complement of an infinite (open) cluster, so the existence of a shielded percolation implies an infinite connected path in the complement of an infinite open cluster. We define $p_{shield}(d)$ as the critical (shielded) probability where the shielded component percolates (exists), and we study the value of $p_{shield}(d)$. Eventually, we show the p_{shield} is greater than p_c (the standard percolation threshold) for $d \geq 11$, in addition, this result can be reduced to $d \geq 8$ using a numerical value of p_c.

In Chapter 3, the acceptance profile, $a_n(x)$ (the limit of the ratio of the number of invaded bonds with values $(x, x+\epsilon]$ to the number of bonds assigned values in $(x, x+\epsilon]$ by time n as ϵ approaches to zero) in invasion percolation (see definitions in Chapter 1 and the introduction of Chapter 3), is introduced. Chayes, Chayes and Newman showed $\lim\limits_{n\to\infty} a(x) = 1$ if $x < p_c$ and $\lim\limits_{n\to\infty} a(x) = 0$ if $x > p_c$. However the limit of the acceptance profile at the critical threshold, i.e., $\lim\limits_{n\to\infty} a_n(p_c)$, remains unknown and this work, we study it on two dimensions. This dissertation shows $\lim\limits_{n\to\infty} a_n(p_c)$ is not trivial in two dimensions. i.e., $0 < \liminf\limits_{n\to\infty} a_n(p_c) \leq \limsup\limits_{n\to\infty} a_n(p_c) < 1$.

CHAPTER 1

INTRODUCTION AND BACKGROUND

Percolation theory was initiated by Broadbent and Hammersley in (1957) [1] , and many mathematicans have studied it. There are many models in percolation theory, but here we introduce the Bernoulli bond percolation and invasion percolation. Let $\mathbb{Z}^d$ be the d-dimensional cubic lattice, and let the vertices be all vectors with integer coordinates. Define the edge set $\mathcal{E}^d = \{< x, y >: x, y \in \mathbb{Z}^d, ||x - y||_{L^1} = 1\}$, where $|| \cdot ||_{L^1}$ is L^1-norm. As a sample space, we take $\Omega = \prod_{e \in \mathcal{E}^d} \{0, 1\}$, and take $\mathcal{F}$ to be σ-field of subsets of Ω generated by the finite dimensional cylinders. Finally, we take the product measure with $\mathbb{P}_p(\omega(e) = 1) = p$ and $\mathbb{P}_p(\omega(e) = 0) = 1 - p$. In bond percolation, we call the edge e open if $\omega(e) = 1$, and closed otherwise. Denote the connected components of open edges as open clusters. One of the main goals in percolation theory is to determine the existence of an infinite open cluster. Let $C(\mathbf{0})$ be the open cluster containing the origin, and define the percolation probability as

$$\begin{aligned} \theta(p) &:= \mathbb{P}(|C(\mathbf{0})| = \infty) \\ &= \mathbb{P}_p(\text{the origin is in an infinite open cluster}). \end{aligned} \tag{1.0.1}$$

As p increases, more open edges exist; therefore $\theta(p)$ is a non-decreasing function as p (See Figure 1.1). Since $\theta(p)$ is non-decreasing as p, $\theta(0) = 0$, and $\theta(1) = 1$, there exists a critical point in $[0, 1]$ so that $\theta(p)$ is positive for p larger than it. We define such a critical point as

$$p_c := \inf\{p \in [0, 1] : \theta(p) > 0\}. \tag{1.0.2}$$

By the ergodic theorem, $\theta(p) > 0$ if and only if there exists almost surely an infinite open cluster; i.e., there does not a.s. exist an infinite open cluster if $p < p_c$, but there

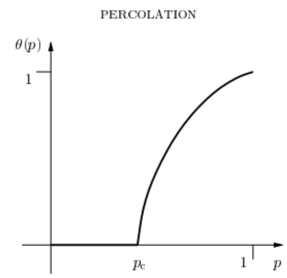

Figure 1.1: The critical probability.

exists an infinite open cluster if $p > p_c$. Then, how many infinite open clusters exist when $p > p_c$? Surprisingly, the number of an infinite cluster is at most one; i.e., there exists a.s. unique infinite open cluster when $p > p_c$ [2, 3]. How about the existence of infinite open clusters at p_c? The question is still an open problem, but at least the following are proved: on the cubic lattice, $\theta(p_c) = 0$ when $d = 2$ (by Kesten [4]) or for any d for which the "triangle condition" holds (it is shown to hold for $d \geq 11$ in [5]). We directly do not use the triangle condition; however, we introduce it because it is an important and remarkable condition for studying the behavior of percolation near the critical threshold [6]. For example, the triangle condition guarantees the existence of mean-field values of various critical exponents, such as $\gamma = 1, \beta = 1, \delta = 2$, where these exponents are roughly defined as

$$\theta(p) \sim (p - p_c)^{\beta} \quad \text{as } p \downarrow p_c$$

$$\chi(p) \sim (p_c - p)^{-\gamma} \quad \text{as } p \uparrow p_c$$

$$\mathbb{P}_{p_c}(|C(0)| \geq n) \sim n^{-1/\delta} \quad \text{as } n \to \infty.$$

Here, $\chi(p) = \mathbb{E}_p|C(\mathbf{0})|$ (the expected size of the open cluster of the origin) and $a_n \sim b_n$ as $n \to \infty$ means $\frac{a_n}{b_n} \to 1$ as $n \to \infty$.

The triangle condition is as follows.

$$\textit{Triangle condition :} \sum_{x,y\in\mathbb{Z}^d} \mathbb{P}_{p_c}(\mathbf{0} \leftrightarrow x)\mathbb{P}_{p_c}(x \leftrightarrow y)\mathbb{P}_{p_c}(y \leftrightarrow \mathbf{0}) < \infty. \tag{1.0.3}$$

Most mathematicians who study percolation theory believe it holds for d larger than some critical dimension d_c (which should be 6 ,[6]); nevertheless it is proved only for $d \geq 11$ in [7].

Let us look over the behavior of $p_c(d)$. Since the d-dimensional lattice is a subgraph of the $(d+1)$-dimensional lattice, it is not difficult to show that $p_c(d)$ is decreasing as d. In addition, it is well-known that $p_c \in (0,1)$ in [8] for all d (in particular, $\frac{1}{2d-1} \leq p_c(d) \leq \frac{1}{2}$ for all d). Many mathematicians have tried to find the exact value of p_c, and Kesten showed $p_c = \frac{1}{2}$ when $d = 2$ in [4]. Also, $p_c(d) \sim \frac{1}{2d}$ as $d \to \infty$ was proved by Kesten [9] and Gordon [10].

Next, we introduce site percolation in which we designate each vertex of the lattice $\mathbb{Z}^d$ open with probability p and closed otherwise independently. The open cluster containing a vertex v is the set of vertices connected to v by a path all of whose vertices are open. Similar to p_c in bond percolation, we define the critical site percolation probability as

$$p_c^{site} := \inf\{p \in [0,1] : \mathbb{P}(|C^{site}(\mathbf{0})| = \infty) > 0\}, \tag{1.0.4}$$

where $C^{site}(0)$ is the open cluster containing the origin in site percolation.

To avoid confusion, we denote the critical bond percolation probability by p_c and the critical site percolation probability by p_c^{site}. One may ask about the relationship between p_c and p_c^{site}. Generally, $p_c \geq p_c^{site}$ for all d and for any lattices.

Now, we introduce some useful formulas in percolation theory. We say the event A is increasing if $I_A(\omega) \leq I_A(\omega')$ whenever $\omega, \omega' \in \Omega$ satisfy $\omega \leq \omega'$ (that is, $\omega(e) \leq \omega'(e)$ for all $e \in \mathcal{E}^d$). In addition, we call the event A decreasing if $I_A(\omega) \geq I_A(\omega')$ for all ω, ω' with

$\omega \leq \omega'$.

(Fortuin–Kasteleyn–Ginibre (FKG) inequality [8, section 2]). Suppose $\mathbb{P}$ is a product measure. If the events A and B are increasing events (or both are decreasing events), then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B). \tag{1.0.5}$$

Moreover, for increasing events (or all decreasing events) A_i, we have

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq \prod_{i=1}^{n} \mathbb{P}(A_i).$$

The FKG inequality is frequently used to find lower bounds for probabilities in percolation theory. For example, we consider the event $\{$there exists an infinite open cluster$\}$ (The open cluster does not need to contain the origin). For any $x \in \mathbb{Z}^d$, we define

$$\begin{aligned}
&p_c(x) := \inf\{p \in [0,1] : \mathbb{P}_p(|C(x)| = \infty)\} \\
&p_c = p_c(\mathbf{0}).
\end{aligned}$$

Suppose there exists $x \in \mathbb{Z}^d$ so that $p_c(x) \geq p_c > 0$. The event $\{|C(x)| = \infty\}$ is implied by $\{x \leftrightarrow \mathbf{0}\} \cap \{|C(\mathbf{0})| = \infty\}$, where $A \leftrightarrow B$ means there exists a open path from A to B. Since $\{y \leftrightarrow x\}$ and $\{|C(\mathbf{0}| = \infty\}$ are increasing, by the FKG inequality we obtain

$$\begin{aligned}
\mathbb{P}_p\left(|C(x)| = \infty\right) &\geq \mathbb{P}_p\left(\{x \leftrightarrow \mathbf{0}\} \cap \{|C(\mathbf{0})| = \infty\}\right) \\
&\geq \mathbb{P}_p\left(\{x \leftrightarrow \mathbf{0}\}\right) \mathbb{P}_p\left(|C(\mathbf{0})| = \infty\right) \\
&= \mathbb{P}_p\left(x \leftrightarrow \mathbf{0}\right) \theta(p).
\end{aligned}$$

$\mathbb{P}_p(x \leftrightarrow \mathbf{0}) > 0$ whenever $p > 0$. Therefore, for $0 < p < p_c(x)$, one obtains $\theta(p) = 0$. This implies $p_c(x) \leq p_c$ and $p_c = p_c(x)$. For the other case that $p_c(x) \leq p_c$ for some x, we have

$$\theta(p) \geq \mathbb{P}_p\left(|C(x)| = \infty\right)\mathbb{P}_p\left(x \leftrightarrow \mathbf{0}\right).$$

So, $p < p_c$ implies $p < p_c(x)$, and we conclude $p_c(x) = p_c$ for any x. Hence, $p_c(x)$ does not depend on the choice of $x \in \mathbb{Z}^d$.

In addition, FKG can be used to find an upper bound of the probability for combinations of increasing and decreasing events. Since the complement of the decreasing event is increasing event, for an increasing event A and decreasing event B,

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A) - \mathbb{P}(A \cap B^c) \leq \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B^c) \\ &= \mathbb{P}(A)\left(1 - \mathbb{P}(B^c)\right) = \mathbb{P}(A)\mathbb{P}(B). \end{aligned} \tag{1.0.6}$$

As we've seen, the FKG inequality is good tool to estimate lower bounds for increasing (or decreasing) events. How about the event consisted of the increasing and decreasing events? Do we find the lower bound of the probability mixing the increasing and decreasing events? The answer is yes sometimes.

(Harris-FKG inequality (Generalized FKG inequality) [11, Lemma 13]). Suppose A^+, $\tilde{A}^+$ are two increasing events, and A^-, $\tilde{A}^-$ are two decreasing events. Assume that there exist three disjoint finite sets of vertices A, A^+ and A^- such that A^+, A^-, $\tilde{A}^+$ and $\tilde{A}^-$ depend only on the sites in, respectively, $A \cup A^+$, $A \cup A^-$, A^+ and A^-. Then for any product measure $\mathbb{P}$, we have

$$\mathbb{P}(\tilde{A}^+ \cap \tilde{A}^- | A^+ \cap A^-) \geq \mathbb{P}(\tilde{A}^+)\mathbb{P}(\tilde{A}^-). \tag{1.0.7}$$

Another formula to find the upper bound probability in percolation theory is the BK inequality. First, we introduce disjoint occurrence.

Suppose that A and B are increasing events of depending on an edge set $E \subseteq \mathcal{E}^d$. Define

$$A \circ B = \{\omega \in \Omega | \text{ there is an } F \subseteq E \text{ so that } \omega_{E \setminus F} \in A \text{ and } \omega_F \in B \}, \tag{1.0.8}$$

where ω_A is the configuration on only set A. That means, the increasing events A and B depend on disjoint set.

(van den Berg-Kesten inequality (BK inequality) [8, section 2]). Let $\mathbb{P}$ be a product measure. Suppose that the edge-set $E \subset \mathcal{E}^d$ is finite, and suppose moreover that A and B are increasing events. Then

$$\mathbb{P}(A \circ B) \leq \mathbb{P}(A)\mathbb{P}(B). \tag{1.0.9}$$

For example, we consider the probability of $\{\mathbf{0} \leftrightarrow \partial B(nk)\}$ for large $n \geq 1$ and fixed k. Here $B(k)$ is the box with centered the origin and the sides $2k$ and $\partial B(k)$ means the boundary of $B(k)$. If the event $\{\mathbf{0} \leftrightarrow \partial B(2k)\}$ occurs, there exists $x \in \partial B(k)$ so that the events $\{\mathbf{0} \leftrightarrow x\}$ and $\{x \leftrightarrow \partial B(2k)\}$ occur disjointly. From the BK inequality,

$$\mathbb{P}\left(\mathbf{0} \leftrightarrow \partial B(2k)\right) \leq \sum_{x \in \partial B(k)} \mathbb{P}\left(\mathbf{0} \leftrightarrow x\right) \mathbb{P}\left(x \leftrightarrow \partial B(k,x)\right),$$

where $B(k,x)$ is the box with center x and sidelength $2k$.

By definition, $\mathbb{E}_p(N_k) = \sum_{x \in \partial B(k)} \mathbb{P}\left(\mathbf{0} \leftrightarrow \partial B(k)\right)$. By translation invariance, we get

$$\mathbb{P}\left(\mathbf{0} \leftrightarrow \partial B(2k)\right) \leq \mathbb{E}_p(N_k)\mathbb{P}\left(\mathbf{0} \leftrightarrow \partial B(k)\right),$$

By a similar argument, we conclude that

$$\mathbb{P}\left(\mathbf{0} \leftrightarrow \partial B(nk)\right) \leq \left(\mathbb{E}_p(N_k)\right)^n.$$

Since $\mathbb{E}_p(N_k) = \mathbb{P}(\mathbf{0} \leftrightarrow \partial B(k)) \xrightarrow{k\to\infty} 0$ in $p < p_c$ [12], we assume $\mathbb{E}_p(N_k) < 1$ for some fixed k. Then, on $p < p_c$, there exists $\sigma(p) = -\ln \mathbb{E}N_k > 0$ so that

$$\mathbb{P}(\mathbf{0} \leftrightarrow \partial B(nk)) \leq e^{-n\sigma(p)}.$$

This means the radius of an open cluster has exponential decay when $p < p_c$.

Now, let's turn to the another percolation model, invasion percolation. Let $\mathbb{Z}^2$ be the two-dimensional square lattice and $\mathcal{E}^2$ be the set of nearest-neighbor edges. For a subgraph $G = (V, E)$ of $(\mathbb{Z}^2, \mathcal{E}^2)$, we define the outer (edge) boundary of G as

$$\partial G := \{e = \{x, y\} \in \mathcal{E}^d : e \notin E, \text{but } x \in G \text{ or } y \in G\}.$$

Assign i.i.d uniform random $[0, 1]$ variables $(\omega(e))$ to all bonds $e \in \mathcal{E}^2$. The *invasion percolation cluster* (IPC) G can be defined as the limit of an increasing sequence of subgraphs (G_n) as follows. The graph G_0 has only the origin and no edges. Once $G_i = (V_i, E_i)$ is defined, we select the edge e_{i+1} that minimizes $\omega(e)$ for $e \in \partial G_i$, take $E_{i+1} = E_i \cup \{e_{i+1}\}$ and let G_{i+1} be the graph induced by the edge set E_{i+1}. Let $V_\infty = \cup_{i=0}^\infty$ and $E_\infty = \cup_{i=0}^\infty$. The graph G_i is called the invaded region at time i, and the graph $G = (V_\infty, E_\infty)$ is called the *invasion percolation cluster* (IPC). For each bond $e \in \mathcal{E}^d$ in the IPC, we call e p-open if $\omega(e) < p$ and p-closed if $\omega(e) > p$. Then the set of p-open edges in the IPC has same distribution as the set of open edges in Bernoulli bond percolation at a density p. So, there is a.s. no infinite p-open cluster in IPC when $p < p_c$ and there exists a.s. one infinite p-open cluster for any $p > p_c$, where p_c is the usual critical probability for Bernoulli bond percolation. Furthermore, after the IPC touches a p-open cluster for any $p > p_c$, it will never escape it. From this fact, we can derive $\limsup_{i\to\infty} \omega(e_i) = p_c$ if e_i is the invaded edge at step i. Since $\theta(p_c) = 0$ in the case $d = 2$, there exist infinitely many edges whose values are greater than p_c. The last two results give that $\hat{\omega}_1 = \max\{\omega(e) : e \in E_\infty\}$ exists and $\hat{\omega}_1 > p_c$. We write the edge whose value is $\hat{\omega}_1$ as $\hat{e}_1$ and we call it the *first outlet*. Let

$\hat{\omega}_2 = \max\{\omega(e_i) : e_i \in E_\infty, i > i_1\}$ and $\hat{e}_2$ be the edge which attains the value $\hat{\omega}_2$. We call $\hat{e}_2$ the *second outlet*. From the previous facts, there exist infinitely many outlets and all outlets' weights are greater than p_c in $d = 2$. In this paper, we do not directly use the properties of the *outlet*, but this terminology is important to understand invasion percolation (See [13]).

Instead, in this dissertation we study the "acceptance profile" of the invasion: the acceptance profile $a_n(p)$ at value p and time n is defined as the ratio

$$a_n(p) = \frac{\text{expected number of bonds invaded with weight in } [p, p + \mathrm{d}p]}{\text{expected number of bonds observed with weight in } [p, p + \mathrm{d}p]},$$

where both the numerator and denominator are computed until time n, and a bond is observed by time n if it is either invaded by time n or is on the boundary of the invasion at time n. (the formal definition in section 3.1.2). This terminology was studied by Chayes, Chayes and Newman in [14], and they showed that for general dimensions, if $p < \pi_c$ (a certain critical threshold for independent percolation), one has $a_n(p) \to 1$ as $n \to \infty$ and if $p > \bar{p}_c$ (another threshold value with $\bar{p}_c \geq \pi_c$), one has $a_n(p) \to 0$ as $n \to \infty$. Since publication of that paper, it has been well-established that $\bar{p}_c = \pi_c = p_c$, where p_c is the standard critical value for independent percolation. Since $p_c = 1/2$ in dimension 2, we have

$$\lim_{n\to\infty} a_n(p) = \begin{cases} 1 & \text{if } p < 1/2 \\ 0 & \text{if } p > 1/2. \end{cases}$$

The case $p = p_c$ was left open in [14], and it is this case we study here. It would be very interesting to establish the existence of $\lim_{n\to\infty} a_n(p_c)$; our result implies that this number would be in $(0, 1)$.

In this dissertation, first we study geometric properties of infinite cluster in Chapter 2. Chapter 2.1 shows the main question and its corresponding notations such as p_{fin}, p_{shield}, and then the main result is in Chapter 2.2. We prove the main result in the rest of Chapter

2, and we show numerical results in Chapter 2.6. Some contents, notations and explanations in Chapter 2 may come from the paper "*Percolation of finite clusters and existence of infinite shielded paths*" by Bounghun Bock and Michael Damron and Charles M. Newman and Vladas Sidoravicius in [15]. Chapter 3 talks about the acceptance profile of invasion percolation. From $\theta(p_c) = 0$ in bond Bernoulli percolation on the case $d = 2$, one may guess $\lim\limits_{n \to p_c} a_n(p_c) = 1$. However, Bernoulli bond percolation and invasion percolation have different features, so we must approach it in a different way. To understand the invasion percolation, we introduce useful notation, formulas and properties in Chapter 3.1 and Chapter 3.2. Last, we prove $\liminf_{n\to\infty} a_n(p_c) > 0$ and $\limsup_{n\to\infty} a_n(p_c) < 1$ on $d = 2$. Unfortunately, the existence of the limit of the acceptance profile and the behavior of acceptance profile on other dimensions ($d \geq 3$) are still open questions. The contents in Chapter 3 may come from the paper "*The acceptance profile of invasion percolation at* p_c *in two dimensions*" by Bounghun Bock and Michael Damron in [16].

CHAPTER 2

PERCOLATION OF FINITE CLUSTERS AND EXISTENCE OF INFINITE SHIELDED PATHS

The structure, contents, notations and explaination in Chapter 2 are derived from [15].

2.1 Introduction

A question for super critical phase(i.e., $p > p_c$) is the geometric properties of infinite clusters. The uniqueness of the infinite open cluster was proved by Aizenman, Kesten and Newman in [2], and Burton and Keane showed it with simple but beautiful idea [8, Theorem 8.1]; a.s., there is a unique infinite cluster if $p > p_c$. In this chapter, following Grimmett-Holroyd-Kozma [7], we study the complement of the infinite cluster. Let X be the subgraph of $\mathbb{Z}^d$ obtained after removing all vertices in the infinite cluster. The complementary critical value, p_{fin}, is defined as

$$p_{fin} = p_{fin}(d) = \sup\{p \in [0,1] : \mathbb{P}_p(X \text{ has an infinite connected component}) > 0\}.$$

In dimension $d = 2$, it is known that $p_c = 1/2$ [4] and that for each $p > p_c$, the infinite cluster contains infinitely many circuits (paths whose initial and final points coincide) around the origin. This implies that $p_{fin}(2) \leq p_c(2)$. Because the definition of $p_{fin}(d)$ implies

$$p_{fin}(d) \geq p_c(d) \text{ for all } d, \tag{2.1.1}$$

we obtain $p_{fin}(2) = 1/2$.

Due to (2.1.1), we have the following question: which d do satisfy $p_{fin}(d) > p_c(d)$? It is natural to believe that this is true for large d because $\theta(p_c) = 0$ [8, Section 10.3] and so for $p = p_c + \epsilon$ and $\epsilon > 0$ small, one expects an infinite cluster with small asymptotic density whose removal is likely to leave much of $\mathbb{Z}^d$ intact. The inequality $p_c(d) < p_{fin}(d)$ for $d \geq 19$ was proved by Grimmett-Holroyd-Kozma in [7] using the triangle condition [6]. Later, Fitzner-van der Hofstad verified the triangle condition for $d \geq 11$ [5], so

$$p_c(d) < p_{fin}(d) \text{ for } d \geq 11. \tag{2.1.2}$$

We will develop a different approach to p_{fin} with "shielded percolation." Let the vertex x shielded if all edges incident to x are closed. A path whose vertices are shielded is called a shielded path. We define the shielded critical probability as

$$p_{shield} := \sup\{p \in [0,1] : \mathbb{P}_p(\exists \text{ an infinite shielded path}) > 0\}$$

Contrary to the critical probability p_c, there a.s. exists an infinite shielded path if $p < p_{shield}$. Furthermore, by the definition of the critical shielded probability, if $p < p_{shield}$, then there exists an infinite connected component in X. Thus, for any d,

$$p_{shield}(d) \leq p_{fin}(d). \tag{2.1.3}$$

If we can find the lower bounds of p_{shield}, we therefore obtain them for p_{fin}. In this chapter, I reprove the Grimmett-Holroyd-Kozma result (2.1.2) using shielded percolation. Furthermore, using numerical values of p_c from [17, 18], we will also verify that (2.1.2) should hold for all $d \geq 8$.

Before showing the main results, I'll explain the idea for proving lower bounds for p_{shield} in briefly. To show it, I used the idea of Cox-Durrett [19], in their study on the asymptotics of the threshold for oriented percolation. One shows that for certain values of p, the number of open oriented paths from 0 to distant hyperplanes has uniformly positive mean, and suitably bounded second moment. The Paley-Zygmund inequality then implies that there are oriented infinite clusters for such p. In running a version of this argument for shielded paths, we obtain the existence of infinite oriented shielded paths for certain values of p. Because the oriented shielded value is smaller than p_{shield}, it is conceivable that more sophisticated lower bounds for p_{shield} would allow to reduce the dimensions (11 and 8) in our results.

2.2 Main results

First, we start with an upper bound for p_{shield}. Let $\lambda(d)$ be the connective constant for vertex self-avoiding walks on $\mathbb{Z}^d$. It is defined as

$$\lambda(d) = \lim_{n\to\infty} \left(\#\{\text{vertex self-avoiding paths with } n \text{ vertices, started at } 0\}\right)^{1/n}.$$

Theorem 2.2.1. *For any $d \geq 1$,*

$$p_{shield}(d) \leq 1 - \lambda(d)^{-\frac{1}{2d-1}}.$$

Using the elementary bound $\lambda(d) \leq 2d - 1$, Theorem 2.2.1 implies

$$p_{shield}(d) \leq 1 - \left(\frac{1}{2d-1}\right)^{\frac{1}{2d-1}}. \tag{2.2.1}$$

Therefore $p_{shield}(2) \leq 1 - \left(\frac{1}{3}\right)^{\frac{1}{3}} \sim 0.306... < \frac{1}{2}$, and we obtain

$$p_{shield}(2) < p_{fin}(2) = p_c(2).$$

(For $d = 3$, we obtain $p_{shield}(3) \leq 1 - \left(\frac{1}{5}\right)^{\frac{1}{5}} \sim 0.275...$, which is larger than $p_c(3) \sim 0.248....$) In contrast, the next result implies that $p_{shield}(d) > p_c(d)$ for large d.

Write $\mathbf{e}_i$ for the i-th standard basis vector, and let $(X_n), (X'_n)$ be i.i.d. with $\mathbb{P}(X_n = \mathbf{e}_i) = \mathbb{P}(X'_n = \mathbf{e}_i) = \frac{1}{d}$ for $1 \leq i \leq d$. S_n, S'_n are defined as the sum of the first n terms respectively with $S_0 = S'_0 = 0$. Define the return probability

$$p_2 = \mathbb{P}(\|S_n - S'_n\|_1 = 2 \text{ for some } n \geq 2 \mid \|S_1 - S'_1\|_1 = 2).$$

Theorem 2.2.2. *Suppose that $d \geq 4$ and that p satisfies the conditions*

1. $p < 1 - \left(\frac{1}{d}\right)^{\frac{1}{2d-1}}$ *and*

2. $\frac{1}{(1-p)^2}\left(p_2 - \frac{1}{d^2} + \frac{1}{d}\left(1 - \frac{1}{d}\right)\left(d(1-p)^{2d-1} - 1\right)^{-1}\right) < 1.$

Then $p_{shield}(d) \geq p$.

The previous result states that p_{shield} can be bounded in terms of the return probability p_2. It is difficult to find the exact value of p_2, but at least we can calculate bounds for it. As a result of above theorems, we get the following corollaries. Write $a_n \sim b_n$ for real sequences (a_n) and (b_n) if $a_n/b_n \to 1$ as $n \to \infty$.

Corollary 2.2.3.

$$p_{shield}(d) \sim \frac{\log d}{2d} \text{ as } d \to \infty. \tag{2.2.2}$$

Remark 2.2.4. *Corollary 2.2.3 implies that*

$$\liminf_{d \to \infty} \frac{p_{fin}(d)}{\frac{\log d}{2d}} \geq 1.$$

It would be interesting to have asymptotic upper bounds for $p_{fin}(d)$. Is $\frac{\log d}{2d}$ the correct order of p_{fin}, as it is [7] on the $2d$-regular tree?

Corollary 2.2.5. *For $d \geq 11$,*

$$p_c(d) < p_{shield}(d) \leq p_{fin}(d). \tag{2.2.3}$$

If numerical values of $p_c(d)$ from [17] or [18] are used, we can improve the dimension in Corollary 2.2.5 to $d \geq 8$. This is shown in Table 2.2 in the appendix.

2.3 Short proof of $p_{shield}(d) > p_c(d)$ for large d

In this section we give a short proof that $p_{shield}(d)$ (and therefore $p_{fin}(d)$) is larger than $p_c(d)$ if d is large. Let $a > 1$ and fix d_* so that

$$p_c^{site}(d_*) < e^{-a}. \tag{2.3.1}$$

(This is possible since $p_c^{site}(d) \to 0$ as $d \to \infty$.) For $d \geq d_*$, set

$$\mathbb{Z}_*^d = \{x \in \mathbb{Z}^d : x \cdot \mathbf{e}_i = 0 \text{ for } i = d_* + 1, \dots, d\}.$$

We say that a vertex $x \in \mathbb{Z}_*^d$ is partially shielded if all edges of the form $\{x, x \pm \mathbf{e}_i\}$ are closed for $i = d_* + 1, \dots, d$. Note that the partially shielded vertices form an independent site percolation process on $\mathbb{Z}_*^d$ with parameter $(1-p)^{2(d-d_*)}$. Set $p = p(d) = \frac{a}{2d}$ so that, because $p_c(d) \sim \frac{1}{2d}$, we have $p > p_c$ for large d. Furthermore, for any $x \in \mathbb{Z}_*^d$,

$$\mathbb{P}_p(x \text{ is partially shielded}) = \left(1 - \frac{a}{2d}\right)^{2(d-d_*)} \to e^{-a} \text{ as } d \to \infty.$$

For $x \in \mathbb{Z}_*^d$, we define Y_x to be the indicator of the event that all edges of the form $\{x, x \pm \mathbf{e}_i\}$ are closed for $i = 1, \dots, d_*$. Then the Y_x's form a 1-dependent site percolation process on $\mathbb{Z}_*^d$ (independent of the process of partially shielded vertices) such that for any $x \in \mathbb{Z}_*^d$,

$$\mathbb{P}_p(Y_x = 1) = \left(1 - \frac{a}{2d}\right)^{2d_*} \to 1 \text{ as } d \to \infty.$$

Therefore the result of Liggett-Schonmann-Stacey [8, Theorem 7.65] implies that (Y_x) is stochastically bounded below by an independent site percolation process (Z_x) with $\mathbb{P}(Z_x =$

$1) \to 1$ as $d \to \infty$. We will assume that the variables Z_x are coupled with the original percolation process so that if $Z_x = 1$, then $Y_x = 1$ and that the Z_x's are independent of the process of partially shielded vertices.

Call $x \in \mathbb{Z}_*^d$ green if x is partially shielded and $Z_x = 1$. Then the set of shielded vertices in $\mathbb{Z}_*^d$ contains the set of green vertices. Since

$$\mathbb{P}_p(x \text{ is green}) = \left(1 - \frac{a}{2d}\right)^{2(d-d_*)} \mathbb{P}(Z_x = 1) \to e^{-a} \text{ as } d \to \infty,$$

inequality (2.3.1) implies that for d large, this probability is $> p_c^{site}(d_*)$. Because the green sites form an independent site percolation process on $\mathbb{Z}_*^d$, one has

$$\mathbb{P}_p(\text{there is an infinite component of green vertices}) > 0 \text{ for large } d.$$

This implies that for large d, one has $p_{shield}(d) \geq p = \frac{a}{2d} > p_c(d)$.

2.4 Proofs of Theorems 2.2.1 and 2.2.2

Proof of Theorem 2.2.1. Suppose $p < p_{shield}$; that is, there is a.s. an infinite shielded path Γ. This is a path, which we will take to be vertex self-avoiding, whose vertices are all shielded. By translation invariance, the probability that the origin is contained in such a path is positive. We will use a Peierls-type argument to show that p cannot be too large.

To do this, we enumerate the vertices of Γ as $0 = x_0, x_1, \ldots$, and use a type of loop-erasure to produce from them another vertex self-avoiding path with shielded vertices $0 = y_0, y_1, \ldots$. We begin with $y_0 = 0$. We then define k_1 as the last index such that x_{k_1} is adjacent to y_0. We "remove the loop" between x_0 and x_{k_1} by setting $y_1 = x_{k_1}$. Continuing, assuming we have defined $y_0, \ldots, y_j$ and $k_1, \ldots, k_j$ for $j \geq 1$, we let k_{j+1} be the last index such that $x_{k_{j+1}}$ is adjacent to y_j, and set $y_{j+1} = x_{k_{j+1}}$. Note that there is always at least one such vertex because x_{k_j+1} is adjacent to y_j. Therefore the sequence (k_j) is strictly increasing, and the y_j's are all distinct.

Clearly each of the y_k's is shielded. We note that the sequence (y_j) has the following properties:

$$y_j \text{ is adjacent to } y_{j-1} \text{ for } j \geq 1 \tag{2.4.1}$$

and

$$y_j \text{ is not adjacent to any of } y_0, \ldots, y_{j-2} \text{ for } j \geq 2. \tag{2.4.2}$$

Indeed, (2.4.1) holds by the definition of y_j (it is x_{k_j}, which is adjacent to $x_{k_{j-1}} = y_{j-1}$). To see (2.4.2), note that if $i = 0, \ldots, j-2$, then k_{i+1} is the last index such that $x_{k_{i+1}}$ is adjacent to y_i, and since $i+1 \leq j-1$, the number k_j must be strictly larger than k_{i+1}. Therefore y_j cannot be adjacent to y_i.

Let Ξ_n be the set of sequences $0 = y_0, \ldots, y_n$ of distinct vertices with properties (2.4.1) (for $j = 1, \ldots, n$) and (2.4.2) (for $j = 2, \ldots, n$). Then the probability that any $\gamma \in \Xi_n$ is shielded is $q^{2d}(q^{2d-1})^n$, where $q = 1-p$. Because $p < p_{shield}$, for each n,

$$0 < \inf_m \mathbb{P}_p\left(\text{some } \gamma \in \Xi_m \text{ is shielded}\right) \leq \sum_{\gamma \in \Xi_n} q^{2d}(q^{2d-1})^n = q^{2d}\left(q^{n(2d-1)} \# \Xi_n\right).$$

Because of property (2.4.1), each $\gamma \in \Xi_n$ is a vertex self-avoiding path with $n+1$ vertices, started at 0. The number of such paths equals $(\lambda(d) + o(1))^{n+1}$ as $n \to \infty$, so

$$\text{if } p < p_{shield}(d), \text{ then } q^{2d}\left(q^{n(2d-1)}(\lambda(d)+o(1))^{n+1}\right) \text{ is bounded away from 0 as } n \to \infty.$$

This implies $q^{2d-1}\lambda(d) \geq 1$, and so we find $p \leq 1 - \lambda(d)^{-\frac{1}{2d-1}}$ for any p satisfying $p < p_{shield}(d)$. This completes the proof. □

Next, we move to lower bounds for p_{shield}.

Proof of Theorem 2.2.2. We use a version of the second moment method from oriented percolation in [19]. Let R_n be the set of oriented paths from the origin to

$$H_n := \left\{ y \in \mathbb{Z}^d : \sum_{i=1}^{d} (y \cdot \mathbf{e}_i) = n \right\}. \tag{2.4.3}$$

Let N_n be the (random) number of shielded paths in R_n. Then,

$$\begin{aligned} \mathbb{E}_p N_n &= \sum_{\gamma \in R_n} \mathbb{P}_p(\text{all sites in } \gamma \text{ are shielded}) = q^{2d}(dq^{2d-1})^n, \text{ and} \qquad (2.4.4) \\ \mathbb{E}_p N_n^2 &= \sum_{\gamma, \gamma' \in R_n} \mathbb{P}_p(\text{all sites in } \gamma \cup \gamma' \text{ are shielded}), \end{aligned}$$

where $q = 1 - p$. The object is now to find values of p for which $\frac{\mathbb{E}_p N_n^2}{(\mathbb{E}_p N_n)^2}$ is bounded away from infinity. If we do this, then $\mathbb{P}_p(N_n \geq 1) = \mathbb{P}_p(N_n > 0) \geq \frac{(\mathbb{E}_p N_n)^2}{\mathbb{E}_p N_n^2}$ will be bounded away from zero, and there will be an infinite (oriented) shielded path with positive probability. For such values of p, then, we will have $p \leq p_{shield}$, and this produces a lower bound on p_{shield}. In other words,

$$\text{if } \sup_n \frac{\mathbb{E}_p N_n^2}{(\mathbb{E}_p N_n)^2} < \infty, \text{ then } p \leq p_{shield}. \tag{2.4.5}$$

We now write the probability in the sum for $\mathbb{E}_p N_n^2$ as a product of many factors. First, for the path γ, we get a factor $q^{2d} q^{(2d-1)n}$. For the other path, write the vertices of γ (in order) as $x_0, \ldots, x_n$ and the vertices of γ' as $x'_0, \ldots, x'_n$. Let k satisfy $0 \leq k \leq n$. If $x_k = x'_k$, then all edges incident to x'_k have already been counted in the factor for γ. If instead $||x_k - x'_k||_1 = 2$, then x'_k is adjacent to at most two vertices of γ, so we get a factor of at most q^{2d-3}. Last, x'_k is not adjacent to any vertices of γ if $||x_k - x'_k||_1 > 2$, so we get a factor of q^{2d-1}. Let

$$\begin{aligned} Z_n(\gamma, \gamma') &= \#\{k = 1, \ldots, n : x_k = x'_k\}, \text{ and} \\ O_n(\gamma, \gamma') &= \#\{k = 1, \ldots, n : \|x_k - x'_k\|_1 = 2\}. \end{aligned}$$

Then we get the upper bound

$$\begin{aligned}\mathbb{E}_p N_n^2 &\leq \sum_{\gamma,\gamma' \in R_n} q^{2d} q^{(2d-1)n} q^{(2d-3)O_n(\gamma,\gamma')} q^{(2d-1)(n-O_n(\gamma,\gamma')-Z_n(\gamma,\gamma'))} \\ &= \frac{(\mathbb{E}_p N_n)^2}{q^{2d} d^{2n}} \sum_{\gamma,\gamma' \in R_n} q^{-2O_n(\gamma,\gamma')} q^{-(2d-1)Z_n(\gamma,\gamma')}. \end{aligned} \tag{2.4.6}$$

We now represent $O_n(\gamma, \gamma')$ and $Z_n(\gamma, \gamma')$ using random walks. Let $(X_k), (X'_k)$ be i.i.d. sequences with

$$\mathbb{P}(X_k = \mathbf{e}_i) = \mathbb{P}(X'_k = \mathbf{e}_i) = \frac{1}{d} \text{ for } 1 \leq i \leq d.$$

S_n and S'_n are defined as the sum of the first n terms respectively with $S_0 = S'_0 = 0$. Let

$$\begin{aligned} Z_n &= \#\{k = 1, \ldots, n : S_k = S'_k\} \\ O_n &= \#\{k = 1, \ldots, n : ||S_k - S'_k||_1 = 2\}. \end{aligned}$$

Using these variables and (2.4.6), we have the representation

$$\frac{\mathbb{E}_p N_n^2}{(\mathbb{E}_p N_n)^2} \leq \frac{1}{q^{2d}} \mathbb{E} q^{-2O_n} q^{-(2d-1)Z_n},$$

so by the monotone convergence theorem,

$$\frac{\mathbb{E}_p N_n^2}{(\mathbb{E}_p N_n)^2} \leq \frac{1}{q^{2d}} \mathbb{E} q^{-2O-(2d-1)Z} \text{ for all } n,$$

where $Z = \lim_{n \to \infty} Z_n$ and $O = \lim_{n\to\infty} O_n$. Putting this in (2.4.5), we obtain

$$\text{if } \mathbb{E} q^{-2O-(2d-1)Z} < \infty, \text{ then } p \leq p_{shield}. \tag{2.4.7}$$

We compute the expectation in the following lemma. Along with (2.4.7), it immediately implies the main result, Theorem 2.2.2. (The condition $p_2 < 1$ for $d \geq 4$ will be verified in

Lemma 2.5.1.) $\square$

Lemma 2.4.1. *Assume that $p_2 < 1$ and that $q = 1 - p$ satisfies*

$$dq^{2d-1} > 1 \text{ and } f(q) < q^2,$$

where

$$f(q) = \left(\frac{1}{d} - \frac{1}{d^2}\right)\left(dq^{2d-1} - 1\right)^{-1} + p_2 - \frac{1}{d^2}.$$

Then

$$\mathbb{E}q^{-2O-(2d-1)Z} = (1 - p_2)\left(1 - \frac{1}{d}\right)\frac{dq^{2d-1}}{dq^{2d-1} - 1}(q^2 - f(q))^{-1} < \infty.$$

Proof. Let $h_n = ||S_n - S'_n||_1$ for $n \geq 0$, and note that $(S_n - S'_n)_{n\geq 0}$ is a Markov chain on $\mathbb{Z}^d$ started at the origin. The sequence (h_n) takes values in $\{0, 2, \dots\}$, but is not a Markov chain. However, computations give the following probabilities for it:

$$\begin{aligned}
&\mathbb{P}(h_k = 0 \mid h_{k-1} = 0) = \frac{1}{d}, \qquad \mathbb{P}(h_k = 2 \mid h_{k-1} = 0) = 1 - \frac{1}{d}, \text{ for } k \geq 1 \text{ and} \\
&\mathbb{P}(h_k = 2 \mid h_{k-1} = 2) = \frac{3d-4}{d^2}, \qquad \mathbb{P}(h_k = 0 \mid h_{k-1} = 2) = \frac{1}{d^2} \text{ for } k \geq 2.
\end{aligned} \tag{2.4.8}$$

Furthermore, since $p_2 < 1$, the strong Markov property implies $O < \infty$ a.s..

Let $(\mathcal{F}_n)$ be the filtration generated by $(X_k, X'_k : k = 1, \dots, n)$, and define the stopping times

$$\tau_0 = 0, \quad \tau_1 = \inf\{n \geq 1 : h_n = 2\}, \text{ and generally}$$
$$\tau_k = \inf\{n \geq \tau_{k-1} + 1 : h_n = 2\} \text{ for } k \geq 1.$$

We then decompose the value of Z according to "excursions" from the set $\{h_n = 2\}$. In other words, on the event $\{O = k\}$ for $k \geq 1$, we can write $Z = Z_1 + \cdots + Z_k$, where

$$Z_i = \#\{n \in [\tau_{i-1} + 1, \tau_i] : h_n = 0\}.$$

(For this decomposition to hold, we need that $\#\{n \geq \tau_k + 1 : h_n = 0\} = 0$. This holds a.s. on $\{O = k\}$, since after time τ_k, the chain must move from $\{h_n = 2\}$ to $\{h_n = 4\}$, and never come back — if it moves to $\{h_n = 0\}$, it will a.s. move back to $\{h_n = 2\}$ eventually by (2.4.8).)

Now we compute the expectation in the lemma iteratively, conditioning on each $\mathcal{F}_{\tau_k}$:

$$\begin{aligned}
&\mathbb{E}q^{-2O-(2d-1)Z} \\
&= \sum_{k=1}^{\infty} \mathbb{E}\left[q^{-2k-(2d-1)(Z_1+\cdots+Z_k)}\mathbf{1}_{\{O=k\}}\right] \\
&= \sum_{k=1}^{\infty} q^{-2k}\mathbb{E}\left[\mathbb{E}\left[q^{-(2d-1)(Z_1+\cdots+Z_k)}\mathbf{1}_{\{\tau_k<\infty,\tau_{k+1}=\infty\}} \mid \mathcal{F}_{\tau_k}\right]\right] \\
&= \sum_{k=1}^{\infty} \left(q^{-2k}\mathbb{E}\left[q^{-(2d-1)(Z_1+\cdots+Z_k)}\mathbf{1}_{\{\tau_k<\infty\}}\right]\mathbb{P}\left(\tau_{k+1}=\infty \mid \mathcal{F}_{\tau_k}\right)\right).
\end{aligned}$$

By the strong Markov property, $\mathbb{P}(\tau_{k+1} = \infty \mid \mathcal{F}_{\tau_k}) = \mathbb{P}(h_n \neq 2 \text{ for all } n \geq 2 \mid S_1 - S_1' = x)$ for some (random) $x = S_{\tau_k} - S'_{\tau_k}$ in the set $\{z \in \mathbb{Z}^d : \|z\|_1 = 2\}$. These x are all of the form $\mathbf{e}_i - \mathbf{e}_j$ with $i \neq j$. By symmetry, these probabilities are the same for all x, and can be written as $\mathbb{P}(h_n \neq 2 \text{ for all } n \geq 2 \mid h_1 = 2) = 1 - p_2$. (This argument is similar to the one that gives that $p_2 < 1$ implies $O < \infty$ a.s., stated below (2.4.8).) Therefore

$$\mathbb{E}q^{-2O-(2d-1)Z} = (1-p_2)\sum_{k=1}^{\infty} q^{-2k}\mathbb{E}\left[q^{-(2d-1)(Z_1+Z_2+\cdots+Z_k)}\mathbf{1}_{\{\tau_k<\infty\}}\right].$$

Now conditioning on $\mathcal{F}_{\tau_{k-1}}$, this equals

$$(1-p_2)\sum_{k=1}^{\infty}\left(q^{-2k}\mathbb{E}\left[q^{-(2d-1)(Z_1+\cdots+Z_{k-1})}\mathbf{1}_{\{\tau_{k-1}<\infty\}}\right]\mathbb{E}\left[q^{-(2d-1)Z_k}\mathbf{1}_{\{\tau_k<\infty\}} \mid \mathcal{F}_{\tau_{k-1}}\right]\right). \tag{2.4.9}$$

As before, by the strong Markov property, the term $\mathbb{E}\left[q^{-(2d-1)Z_k}\mathbf{1}_{\{\tau_k<\infty\}} \mid \mathcal{F}_{\tau_{k-1}}\right]$ for $k \geq 2$ is equal to $\mathbb{E}\left[q^{-(2d-1)Z_2}\mathbf{1}_{\{\tau_2<\infty\}} \mid S_1 - S_1' = x\right]$ for some random $x = S_{\tau_{k-1}} - S'_{\tau_{k-1}}$ of the form $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j$. These expectations are all the same by symmetry, so if we set

$$f(q) = \mathbb{E}\left[q^{-(2d-1)Z_2}\mathbf{1}_{\{\tau_2<\infty\}} \mid h_1 = 2\right],$$

then (2.4.9) gives us

$$\begin{aligned}
&\mathbb{E}q^{-2O-(2d-1)Z} \\
= \ &(1-p_2)\left[q^{-2}\mathbb{E}\left[q^{-(2d-1)Z_1}\mathbf{1}_{\{\tau_1<\infty\}}\right] + \sum_{k=2}^{\infty}\left(q^{-2k}f(q)\mathbb{E}\left[q^{-(2d-1)(Z_1+\cdots+Z_{k-1})}\mathbf{1}_{\{\tau_{k-1}<\infty\}}\right]\right)\right].
\end{aligned}$$

Last, we iterate the above procedure, conditioning successively on $\mathcal{F}_{\tau_{k-1}}, \mathcal{F}_{\tau_{k-2}}, \ldots, \mathcal{F}_{\tau_1}$, to obtain

$$\mathbb{E}q^{-2O-(2d-1)Z} = (1-p_2)\mathbb{E}\left[q^{-(2d-1)Z_1}\mathbf{1}_{\{\tau_1<\infty\}}\right]\sum_{k=1}^{\infty}\left(q^{-2k}f(q)^{k-1}\right),$$

or, because $\tau_1 < \infty$ a.s. (see (2.4.8)),

$$\begin{aligned}
\mathbb{E}q^{-2O-(2d-1)Z} &= (1-p_2)\mathbb{E}q^{-(2d-1)Z_1}\sum_{k=1}^{\infty}\left(q^{-2k}f(q)^{k-1}\right) \\
&= (1-p_2)\mathbb{E}q^{-(2d-1)Z_1}(q^2 - f(q))^{-1} \text{ if } f(q) < q^2. \qquad (2.4.10)
\end{aligned}$$

We now set out to compute the terms in (2.4.10). Beginning with $f(q)$, because $h_2 = 0$ almost surely implies $\tau_2 < \infty$, we obtain

$$\begin{aligned}
f(q) &= \mathbb{E}\left[q^{-(2d-1)Z_2}\mathbf{1}_{\{\tau_2<\infty, h_2=0\}} \mid h_1 = 2\right] + \mathbb{P}\left(\tau_2 < \infty \text{ and } h_2 \neq 0 \mid h_1 = 2\right) \\
&= \mathbb{E}\left[q^{-(2d-1)Z_2}\mathbf{1}_{\{h_2=0\}} \mid h_1 = 2\right] + p_2 - \frac{1}{d^2}. \qquad (2.4.11)
\end{aligned}$$

Furthermore, using (2.4.8), the first term of (2.4.11) equals

$$\begin{aligned}
&\frac{1}{d^2}\mathbb{E}\left[q^{-(2d-1)Z_2} \mid h_1 = 2, h_2 = 0\right] \\
= &\ \frac{1}{d^2}\sum_{j=1}^{\infty} q^{-(2d-1)j}\mathbb{P}\left(h_2 = \cdots = h_{j+1} = 0, h_{j+2} = 2 \mid h_1 = 2, h_2 = 0\right) \\
= &\ \frac{1}{d^2}\sum_{j=1}^{\infty} q^{-(2d-1)j}\left(\frac{1}{d}\right)^{j-1}\left(1-\frac{1}{d}\right) \\
= &\ \frac{1}{d}\left(1-\frac{1}{d}\right)\left(dq^{2d-1}-1\right)^{-1} \text{ if } dq^{2d-1} > 1.
\end{aligned}$$

Putting this in (2.4.11), we obtain

$$f(q) = \left(\frac{1}{d} - \frac{1}{d^2}\right)\left(dq^{2d-1}-1\right)^{-1} + p_2 - \frac{1}{d^2} \text{ when } dq^{2d-1} > 1. \tag{2.4.12}$$

For the other term in (2.4.10), we similarly compute when $dq^{2d-1} > 1$

$$\begin{aligned}
\mathbb{E}q^{-(2d-1)Z_1} &= \sum_{j=0}^{\infty} q^{-(2d-1)j}\mathbb{P}(h_1 = \cdots = h_j = 0, h_{j+1} = 2) \\
&= \left(1-\frac{1}{d}\right)\sum_{j=0}^{\infty} q^{-(2d-1)j}\left(\frac{1}{d}\right)^{j} \\
&= \left(1-\frac{1}{d}\right)\frac{dq^{2d-1}}{dq^{2d-1}-1}.
\end{aligned}$$

We place this and (2.4.12) into (2.4.10) to complete the proof. □

2.5 Proofs of Corollaries 2.2.3 and 2.2.5

We will use the following result in the proofs of both corollaries.

Lemma 2.5.1. *For $d \geq 4$, one has $p_2 < 1$. Furthermore, if we define*

$$p_d = \mathbb{P}(S_n = S_n' \textit{ for some } n \geq 1),$$

then

1. $p_2 = \frac{(d^2+1)p_d - d - 1}{d^2 p_d - d}$, *and*

2.

$$
\begin{aligned}
p_d \leq{}& \frac{1}{d} + \left(1 - \frac{1}{d}\right)\frac{1}{d^2} + \frac{1}{d^2}\left(1 - \frac{1}{d}\right)\left(\frac{3d-4}{d^2}\right) \\
&+ \frac{1}{d^2}\left(\left(\frac{3d-4}{d^2}\right)^2 + \left(\frac{d^2-3d+3}{d^2}\right)\left(\frac{4}{d^2}\right)\right) \\
&+ \sum_{k=5}^{d}\frac{k!}{d^k} + \sum_{j=1}^{\infty}\left(\frac{1}{d}\right)^{jd-1}\frac{(jd)!}{(j!)^d}.
\end{aligned}
$$

Proof. We begin with item 1. We continue with the sequence (h_n) from the previous section, where $h_n = \|S_n - S_n'\|_1$. As before, let

$$Z_n = \#\{k = 1, \ldots, n : h_k = 0\} \text{ and } O_n = \#\{k = 1, \ldots, n : h_k = 2\}.$$

Then, recalling the probabilities in (2.4.8), we compute

$$
\begin{aligned}
\mathbb{E}(1 + Z_n) &= 1 + \sum_{k=1}^{n}\mathbb{P}(h_k = 0) \\
&= 1 + \sum_{k=1}^{n}\Big(\mathbb{P}(h_k = 0, h_{k-1} = 0) + \mathbb{P}(h_k = 0, h_{k-1} = 2)\Big) \\
&= 1 + \sum_{k=1}^{n}\left(\frac{1}{d}\mathbb{P}(h_{k-1} = 0) + \frac{1}{d^2}\mathbb{P}(h_{k-1} = 2)\right) \\
&= 1 + \frac{1}{d}\mathbb{E}(1 + Z_{n-1}) + \frac{1}{d^2}\mathbb{E}O_{n-1}.
\end{aligned}
$$

By the monotone convergence theorem, for $Z = \lim_{n\to\infty} Z_n$ and $O = \lim_{n\to\infty} O_n$, we have

$$\mathbb{E}(1 + Z) = 1 + \frac{1}{d}\mathbb{E}(1 + Z) + \frac{1}{d^2}\mathbb{E}O. \tag{2.5.1}$$

To write (2.5.1) in terms of p_2 and p_d, we note that by the strong Markov property,

$$\mathbb{P}(Z = k) = p_d^k(1 - p_d) \text{ for } k \geq 0, \text{ and} \tag{2.5.2}$$

$$\begin{aligned}\mathbb{P}(O = k) &= (1 - p_2)p_2^{k-1}\mathbb{P}(h_k = 2 \text{ for some } k \geq 1) \\ &= (1 - p_2)p_2^{k-1} \text{ for } k \geq 1. \qquad (2.5.3)\end{aligned}$$

Therefore

$$\mathbb{E}Z = \frac{p_d}{1 - p_d} \text{ and } \mathbb{E}O = \frac{1}{1 - p_2},$$

and (2.5.1) becomes

$$\frac{1}{1 - p_d} = 1 + \frac{1}{d(1 - p_d)} + \frac{1}{d^2(1 - p_2)}.$$

This implies the first item of the lemma.

For the second item, we define the stopping time

$$\tau = \inf\{k \geq 1 : h_k = 0\},$$

so that $p_d = \mathbb{P}(\tau < \infty)$. By a straightforward calculation,

$$\begin{aligned}\mathbb{P}(\tau = 1) &= \frac{1}{d}, \text{ and} \\ \mathbb{P}(\tau = 2) &= \mathbb{P}(h_2 = 0 \mid h_1 = 2)\mathbb{P}(h_1 = 2) \qquad (2.5.4) \\ &= \left(1 - \frac{1}{d}\right)\frac{1}{d^2}.\end{aligned}$$

We will need to compute both $\mathbb{P}(\tau = 3)$ and $\mathbb{P}(\tau = 4)$, and these are a little more complicated. We first claim that

$$\mathbb{P}(\tau = 3) = \frac{1}{d^2}\left(\frac{3d - 4}{d^2}\right)\left(1 - \frac{1}{d}\right). \tag{2.5.5}$$

To show this use (2.4.8) to write

$$
\begin{aligned}
\mathbb{P}(\tau = 3) &= \mathbb{P}(h_1 = 2)\mathbb{P}(h_2 = 2 \mid h_1 = 2)\mathbb{P}(h_3 = 0 \mid h_1 = 2, h_2 = 2) \\
&= \left(1 - \frac{1}{d}\right)\frac{3d-4}{d^2}\mathbb{P}(h_3 = 0 \mid h_1 = 2, h_2 = 2).
\end{aligned}
$$

The last probability is written using the Markov property at time 2 as

$$
\frac{\mathbb{E}\left[\mathbb{P}(h_3 = 0 \mid \mathcal{F}_2)\mathbf{1}_{\{h_1=2,h_2=2\}}\right]}{\mathbb{P}(h_1 = 2, h_2 = 2)} = \frac{\mathbb{E}\left[\mathbb{P}(h_2 = 0 \mid S_1 - S_1' = x)\mathbf{1}_{\{h_1=2,h_2=2\}}\right]}{\mathbb{P}(h_1 = 2, h_2 = 2)},
$$

where x is the (random) value of $S_2 - S_2'$, which must be of the form $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j$. These probabilities are constant as x varies, and are equal to $\frac{1}{d^2}$. Therefore we obtain $\mathbb{P}(h_3 = 0 \mid h_1 = 2, h_2 = 2) = \frac{1}{d^2}$, and this shows (2.5.5).

The situation with $\{\tau = 4\}$ is somewhat worse than that for $\{\tau = 3\}$, and the form is

$$
\mathbb{P}(\tau = 4) \leq \frac{1}{d^2}\left(\left(\frac{3d-4}{d^2}\right)^2 + \left(\frac{d^2-3d+3}{d^2}\right)\left(\frac{4}{d^2}\right)\right)\left(1 - \frac{1}{d}\right). \tag{2.5.6}
$$

The analysis splits into 2 cases:

1. $(h_0, \ldots, h_4) = (0, 2, 2, 2, 0)$,
2. $(h_0, \ldots, h_4) = (0, 2, 4, 2, 0)$.

The first case is computed exactly as we did for $\{\tau = 3\}$: we obtain the form

$$
\left(1 - \frac{1}{d}\right)\left(\frac{3d-4}{d^2}\right)^2\frac{1}{d^2}. \tag{2.5.7}
$$

For the second, we get

$$
\left(1 - \frac{1}{d}\right)\mathbb{P}(h_2 = 4 \mid h_1 = 2)\mathbb{P}(h_3 = 2 \mid h_1 = 2, h_2 = 4)\frac{1}{d^2}.
$$

By (2.4.8),

$$\mathbb{P}(h_2 = 4 \mid h_1 = 2) = 1 - \frac{1}{d^2} - \frac{3d-4}{d^2} = \frac{d^2 - 3d + 3}{d^2},$$

so we obtain

$$\left(1 - \frac{1}{d}\right) \frac{d^2 - 3d + 3}{d^2} \mathbb{P}(h_3 = 2 \mid h_1 = 2, h_2 = 4) \frac{1}{d^2}. \tag{2.5.8}$$

For the other term, we again use the Markov property to write it as

$$\frac{\mathbb{E}\left[\mathbb{P}(h_3 = 2 \mid S_2 - S_2' = x)\mathbf{1}_{\{h_1=2,h_2=4\}}\right]}{\mathbb{P}(h_1 = 2, h_2 = 4)},$$

where x is the (random) value of $S_2 - S_2'$. Up to symmetry, there are 3 different values of x:

(A) $2\mathbf{e}_i - 2\mathbf{e}_j$ for some $i \neq j$,

(B) $2\mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_\ell$ for some distinct i, j, ℓ, and

(C) $\mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_\ell - \mathbf{e}_m$ for some distinct i, j, ℓ, m.

In case (A), X_3 must be $\mathbf{e}_j$ and X_3' must be $\mathbf{e}_i$ to make $h_3 = 2$. This gives a probability of $\frac{1}{d^2}$. In case (B), X_3 must be $\mathbf{e}_j$ or $\mathbf{e}_\ell$ and X_3' must be e_i, giving a probability of $\frac{2}{d^2}$. In case (C), X_3 must be $\mathbf{e}_\ell$ or $\mathbf{e}_m$ and X_3' must be $\mathbf{e}_i$ or $\mathbf{e}_j$, giving a probability of $\frac{4}{d^2}$. In all cases, the probability is bounded above by $\frac{4}{d^2}$. Plugging this into (2.5.8) gives an upper bound of

$$\left(1 - \frac{1}{d}\right) \frac{d^2 - 3d + 3}{d^2} \cdot \frac{4}{d^2} \cdot \frac{1}{d^2}.$$

If we add this to (2.5.7), we obtain the claimed bound in (2.5.6).

For $\mathbb{P}(\tau = k)$ with $k \geq 5$, we use

$$\mathbb{P}(\tau = k) \leq \mathbb{P}(S_k = S_k') \leq \max_{x \in H_k} \mathbb{P}(S_k = x),$$

where we recall that H_k was defined in (2.4.3). Following [19, p. 155], for $1 \leq k \leq d$, the maximum above is attained when $x = \mathbf{e}_1 + \cdots + \mathbf{e}_k$, so

$$\mathbb{P}(\tau = k) \leq \max_{x \in H_k} \mathbb{P}(S_k = x) \leq \frac{k!}{d^k} \text{ for } 1 \leq k \leq d. \tag{2.5.9}$$

To bound $\mathbb{P}(\tau = k)$ for $k > d$, we first claim that $\max_{x \in H_j} \mathbb{P}(S_j = x)$ is nonincreasing in j. Indeed, if this were not true, then we could find j such that $\max_{x \in H_j} \mathbb{P}(S_j = x) > \max_{y \in H_{j-1}} \mathbb{P}(S_{j-1} = y)$. Choosing x corresponding to the maximum in H_j, we could compute

$$\begin{aligned}\mathbb{P}(S_j = x) = \sum_{y \in H_{j-1}} \mathbb{P}(S_{j-1} = y)\mathbb{P}(S_j = x \mid S_{j-1} = y) < \mathbb{P}(S_j = x) \sum_{y \in H_{j-1}} \mathbb{P}(X_j = x - y) \\ = \mathbb{P}(S_j = x),\end{aligned}$$

a contradiction. So, using the claim, if $k \geq jd$ for $j \geq 1$, we estimate, writing $y = (y_1, \ldots, y_d)$,

$$\begin{aligned}\mathbb{P}(\tau = k) \leq \mathbb{P}(S_k = S_k') \leq \max_{x \in H_k} \mathbb{P}(S_k = x) \leq \max_{y \in H_{jd}} \mathbb{P}(S_{jd} = y) = \max_{y \in H_{jd}} \left[\frac{1}{d^{jd}} \cdot \frac{(jd)!}{y_1! \cdot \cdots \cdot y_d!}\right] \\ \leq \left(\frac{1}{d}\right)^{jd} \frac{(jd)!}{(j!)^d}.\end{aligned} \tag{2.5.10}$$

If we write

$$p_d = \mathbb{P}(\tau = 1) + \mathbb{P}(\tau = 2) + \mathbb{P}(\tau = 3) + \mathbb{P}(\tau = 4) + \sum_{k=5}^{d} \mathbb{P}(\tau = k) + \sum_{j=1}^{\infty} \sum_{k=jd+1}^{(j+1)d} \mathbb{P}(\tau = k),$$

and use (2.5.4) for the first and second terms, (2.5.5) for the third, (2.5.6) for the fourth, (2.5.9) for the first sum, and (2.5.10) for the last, we obtain the claimed inequality in item 2.

Finally, we show that for $d \geq 4$, one has $p_2 < 1$. It suffices, in fact, to show that $p_d < 1$

since $p_2 = 1 - \frac{1-p_d}{d^2 p_d - d}$, and if $p_d < 1$ then the numerator is positive (the denominator is always positive since $p_d > \mathbb{P}(h_1 = 0) = \frac{1}{d}$). To show $p_d < 1$, it is enough by (2.5.2) to show that $\mathbb{E}Z < \infty$ and so we estimate as above, using Stirling's approximation with $1 \leq \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \leq e^{\frac{1}{12n}}$ for all $n \geq 1$ from [20, Eq. (9.15)] to obtain

$$\left(\frac{1}{d}\right)^{jd} \frac{(jd)!}{(j!)^d} \leq \sqrt{2\pi d}\frac{e^{\frac{1}{12d}}}{(2\pi)^{\frac{d}{2}}} j^{\frac{1-d}{2}}.$$

and

$$\begin{aligned}\mathbb{E}Z = \sum_{k=1}^{\infty} \mathbb{P}(h_k = 0) \leq \sum_{k=1}^{\infty} \max_{x \in H_k} \mathbb{P}(S_k = x) &\leq \sum_{k=1}^{d} \frac{k!}{d^k} + \sum_{j=1}^{\infty} \sum_{k=jd+1}^{(j+1)d} \frac{(jd)!}{(j!)^d} \\ &\leq \sum_{k=1}^{d} \frac{k!}{d^k} + d\sqrt{2\pi d}\frac{e^{\frac{1}{12d}}}{(2\pi)^{\frac{d}{2}}} \sum_{j=1}^{\infty} j^{\frac{1-d}{2}}.\end{aligned}$$

This is finite for $d \geq 4$. $\square$

Proof of Corollary 2.2.3. From the consequence (2.2.1) of Theorem 2.2.1,

$$\limsup_{d \to \infty} \frac{p_{shield}(d)}{\frac{\log d}{2d}} \leq 1.$$

For the lower bound, we put $p = p(d) = \frac{a \log d}{2d}$ for $a \in (0,1)$, and check that p satisfies the conditions of Theorem 2.2.2. First, $p_2 < 1$ for large d by Lemma 2.5.1. Next, one has

$$\begin{aligned}(1-p)^{2d-1} = \exp\left((2d-1)\log\left(1 - \frac{a\log d}{2d}\right)\right) &= \exp\left(-a(1+o(1))\log d\right) \\ &= d^{-a+o(1)} \text{ as } d \to \infty.\end{aligned}$$

This is $> \frac{1}{d}$ for large d, so the first assumption of Theorem 2.2.2 holds. For the second, the calculation is similar: its left side equals

$$\frac{1}{\left(1 - \frac{a\log d}{2d}\right)^2}\left(p_2 - \frac{1}{d^2} + \frac{1}{d}\left(1 - \frac{1}{d}\right)(d^{1-a+o(1)} - 1)^{-1}\right),$$

which is $p_2(1+o(1))$ as $d \to \infty$. Since $p_2 < 1$, we see this the left side is < 1 for large d, and this verifies item 2. In conclusion, we find that $p_{shield}(d) \geq \frac{a \log d}{2d}$ for large d, whenever $a < 1$, and so

$$\liminf_{d \to \infty} \frac{p_{shield}(d)}{\frac{\log d}{2d}} \geq 1.$$

□

Proof of Corollary 2.2.5. To find values of d for which $p_{shield}(d) > p_c(d)$, we will need a useful upper bound for p_c. Unfortunately, we only have explicit upper bounds for the threshold of oriented percolation. We define the return probability

$$\rho_d = \mathbb{P}(S_k = S'_k,\ S_{k+1} = S_{k'+1} \text{ for some } k \geq 0),$$

and use [19, Eq. (1.1)], which states that the oriented threshold satisfies $\vec{p}_c(d) \leq \rho_d$. Since $p_c(d) \leq \vec{p}_c(d)$, we obtain $p_c(d) \leq \rho_d$.

Define the stopping time $\hat{\tau} = \inf\{k \geq 0 : S_k = S'_k, S_{k+1} = S'_{k+1}\}$, so that $\rho_d = \sum_{k=0}^{\infty} \mathbb{P}(\hat{\tau} = k)$. By similar calculations to those in the proof of Corollary 2.2.3 (the following are listed in [19, p. 155-156]),

$$\mathbb{P}(\hat{\tau} = 0) = \frac{1}{d}, \quad \mathbb{P}(\hat{\tau} = 1) = 0,\ \mathbb{P}(\hat{\tau} = 2) = \frac{1}{d^3} - \frac{1}{d^4},$$

for $l = 1, \ldots, d$,

$$\mathbb{P}(l \leq \hat{\tau} \leq d) \leq \sum_{k=l}^{d} \frac{1}{d} \cdot \frac{k!}{d^k},$$

and

$$\mathbb{P}(d < \hat{\tau} < \infty) \leq \sum_{j=1}^{\infty} \left(\frac{1}{d}\right)^{jd} \frac{(jd)!}{(j!)^d}.$$

We will again want to separate out the cases $\hat{\tau} = k$ for $k = 3, 4$. Doing calculations similar to those in the proof of Lemma 2.5.1, we obtain

$$\mathbb{P}(\hat{\tau} = 3) = \left(1 - \frac{1}{d}\right) \frac{3d-4}{d^2} \cdot \frac{1}{d^3}$$
$$\mathbb{P}(\hat{\tau} = 4) \leq \frac{1}{d^3} \left[\left(\frac{3d-4}{d^2}\right)^2 + \left(\frac{d^2-3d+3}{d^2}\right)\left(\frac{4}{d^2}\right) + \left(\frac{1}{d^2}\right)\left(1-\frac{1}{d}\right)\right]\left(1-\frac{1}{d}\right).$$

(In the first case, the relevant (h_n) vector is $(h_0, \ldots, h_4) = (0, 2, 2, 0, 0)$ and for the second case, they are $(h_0, \ldots, h_5) = (0, 2, 2, 2, 0, 0)$, $(0, 2, 4, 2, 0, 0)$, and $(0, 2, 0, 2, 0, 0)$.)

Combining these estimates and again using Stirling's approximation, we obtain

$$\begin{aligned} p_c \leq \rho_d \leq & \frac{1}{d} + \frac{1}{d^3} - \frac{1}{d^4} + \frac{1}{d^3}\left(\frac{3d-4}{d^2}\right)\left(1-\frac{1}{d}\right) \\ & + \frac{1}{d^3} \left[\left(\frac{3d-4}{d^2}\right)^2 + \left(\frac{d^2-3d+3}{d^2}\right)\left(\frac{4}{d^2}\right) + \left(\frac{1}{d^2}\right)\left(1-\frac{1}{d}\right)\right]\left(1-\frac{1}{d}\right) \\ & + \sum_{k=5}^{d} \frac{k!}{d^{k+1}} + \sqrt{2\pi d}\frac{e^{\frac{1}{12d}}}{(2\pi)^{\frac{d}{2}}} \sum_{j=1}^{\infty} j^{\frac{1-d}{2}} =: g(d) \end{aligned} \tag{2.5.11}$$

To give an explicit lower bound on $p_{shield}(d)$, we will show that for $p = g(d)$, the two conditions of Theorem 2.2.2 hold. That is, we will show that

$$g(d)\left(1 - \left(\frac{1}{d}\right)^{\frac{1}{2d-1}}\right) < 1 \tag{2.5.12}$$

and

$$\frac{1}{(1-g(d))^2}\left(p_2 - \frac{1}{d^2} + \frac{1}{d}\left(1-\frac{1}{d}\right)(d(1-g(d))^{2d-1}-1)^{-1}\right) < 1. \tag{2.5.13}$$

For any d such that these inequalities hold, we must have $p_{shield}(d) > p_c(d)$. Indeed, since the left side of either inequality is a continuous function of $g(d)$, they will also hold for some number $\hat{p} > g(d)$ sufficiently close to $g(d)$, and we will have $p_c \leq g(d) < \hat{p} \leq p_{shield}(d)$.

To show the two inequalities above, we recall Lemma 2.5.1 and the bounds contained therein. From there, we define

$$B(d) = \frac{1}{d} + \left(1 - \frac{1}{d}\right)\frac{1}{d^2} + \frac{1}{d^2}\left(\frac{3d-4}{d^2}\right)\left(1 - \frac{1}{d}\right)$$
$$+ \frac{1}{d^2}\left[\left(\frac{3d-4}{d^2}\right)^2 + \left(\frac{d^2-3d+3}{d^2}\right)\left(\frac{4}{d^2}\right)\right]\left(1 - \frac{1}{d}\right) + \sum_{k=5}^{d}\frac{k!}{d^k} + d\sqrt{2\pi d}\frac{e^{\frac{1}{12d}}}{(2\pi)^{\frac{d}{2}}}\sum_{j=1}^{\infty} j^{\frac{1-d}{2}}$$

and

$$t(x) = \frac{(d^2+1)x - d - 1}{d^2 x - d}.$$

(The function t is defined so that $t(p_d) = p_2$.) Because $t(x) = 1 - \frac{1-x}{d^2x-d}$, it is monotone nondecreasing for $x > 1/d$. Therefore, since $1/d < p_d \leq B(d)$, one has $p_2 \leq t(B(d))$, and we see that it will suffice to show that

$$g(d)\left(1 - \left(\frac{1}{d}\right)^{\frac{1}{2d-1}}\right)^{-1} < 1$$

and

$$\frac{1}{(1-g(d))^2}\left(t(B(d)) - \frac{1}{d^2} + \frac{1}{d}\left(1 - \frac{1}{d}\right)\left(d(1-g(d))^{2d-1} - 1\right)^{-1}\right) < 1.$$

Table 2.1 shows computed values of the left sides of these inequalities. Their values drop below 1 between dimensions 10 and 11.

□

2.6 Numerical results

If we use numerical values of p_c, the result can be reduced to $d = 8$. In other words, we can show that $p_{shield}(d) > p_c(d)$ for $d \geq 8$. The second column of Table 2.2 shows numerical values of $p_c = p_c^{bond}$ for dimensions 5-10. The third column gives lower bounds for $p_{shield}(d)$ for these dimensions. The fourth gives the maximum of the left sides of

Table 2.1: The values of the left sides of (2.5.12) and (2.5.13). The maximum of the two values drops below 1 between $d = 10$ and 11. Because both inequalities hold for $11 \leq d \leq 18$, one has $p_c(d) < p_{shield}(d)$ for these d. (Values computed using Mathematica.)

d	LHS of (2.5.12)	LHS of (2.5.13)
9	0.953 734 5	2.038 305 9
10	0.897 595 0	1.178 003 9
11	0.855 878 5	0.867 982 9
12	0.822 865 5	0.668 997 4
13	0.795 549 3	0.529 081 0
14	0.772 244 9	0.430 773 4
15	0.751 938 7	0.361 897 7
16	0.733 976 5	0.312 899 7
17	0.717 908 0	0.276 975 5
18	0.703 406 0	0.249 603 7

(2.5.12) and (2.5.13) when setting p equal to the value in the third column. Because this maximum is < 1, it shows that the value in the second column is indeed a lower bound for p_{shield}. One can see that the lower bound for p_{shield} is larger than the value of p_c for dimensions 8-10. In Table 2, we used $\hat{B}(d)$ as the upper bound of p_d using (2.5.9), (2.5.10) (without using Stirling's approximation):

$$
\begin{aligned}
p_d \leq \hat{B}(d) := {} & \frac{1}{d} + \left(1 - \frac{1}{d}\right)\frac{1}{d^2} + \frac{1}{d^2}\left(\frac{3d-4}{d^2}\right)\left(1 - \frac{1}{d}\right) \\
& + \frac{1}{d^2}\left[\left(\frac{3d-4}{d^2}\right)^2 + \left(\frac{d^2-3d+3}{d^2}\right)\left(\frac{4}{d^2}\right)\right]\left(1 - \frac{1}{d}\right) + \sum_{k=5}^{d}\frac{k!}{d^k} + \sum_{j=1}^{\infty}\left(\frac{1}{d}\right)^{jd-1}\frac{(jd)!}{(j!)^d},
\end{aligned}
$$

That is, the lower bound of $p_{shield}(d)$ in Table 2.2 is the maximum value of $p(d)$ which satisfies both of the following:

$$p(d)\left(1-\left(\frac{1}{d}\right)^{\frac{1}{2d-1}}\right)^{-1}<1$$

and

$$\frac{1}{(1-p(d))^2}\left(t(\hat{B}(d))-\frac{1}{d^2}+\frac{1}{d}\left(1-\frac{1}{d}\right)\left(d(1-p(d))^{2d-1}-1\right)^{-1}\right)<1.$$

Table 2.2: Numerical values of $p_c = p_c^{bond}$ and lower bounds for p_{shield}. The top numerical value of p_c comes from [17] and the bottom value comes from [18]. The fourth column is the maximum of the left sides of the first and second conditions in Theorem 2.2.2 when p is set equal to the lower bound for p_{shield}, which is the value in the third column. The value in the second column increases above that in the third between $d = 7$ and 8. (Data computed using Mathematica.)

d	p_c^{bond}	lower bound of p_{shield}	max. of left sides in Thm. 2.2.2
5	0.118 171 8 0.118 171 5	0.012726	0.999998
6	0.094 201 9 0.094 201 6	0.034893	0.999997
7	0.078 675 2 0.078 675 2	0.059902	0.999998
8	0.067 708 3 0.067 708 4	0.083526	0.999994
9	0.059 496 0 0.059 496 0	0.097006	0.999993
10	0.053 092 5 0.053 092 5	0.100445	0.999985

CHAPTER 3
THE ACCEPTANCE PROFILE OF INVASION PERCOLATION IN TWO-DIMENSIONS

The structure, contents, notations and explaination in Chapter 3 are derived from [16].

3.1 Introduction

3.1.1 Invasion percolation

We recall the definition of invasion percolation. It is a stochastic growth model introduced independently by two groups [21] and [22], and is a simple example of self-organized criticality. That is, although the model itself has no parameter, its structure on large scales resembles that of another critical model: critical Bernoulli percolation.

Let $\mathbb{Z}^2$ be the two-dimensional square lattice and $\mathcal{E}^2$ be set of nearest-neighbor edges. For a subgraph $G = (V, E)$ of $(\mathbb{Z}^2, \mathcal{E}^2)$, we define the outer (edge) boundary of G as

$$\partial G := \{e = \{x, y\} \in \mathcal{E}^d : e \notin E, \text{but } x \in G \text{ or } y \in G\}$$

Assign i.i.d uniform random $[0, 1]$ variables $(\omega(e))$ to all bonds $e \in \mathcal{E}^2$. The *invasion percolation cluster* (IPC) G can be defined as the limit of an increasing sequence of subgraphs (G_n) as follows. The graph G_0 has only the origin and no edges. Once $G_i = (V_i, E_i)$ is defined, we select the edge e_{i+1} that minimizes $\omega(e)$ for $e \in \partial G_i$, take $E_{i+1} = E_i \cup \{e_{i+1}\}$ and let G_{i+1} be the graph induced by the edge set E_{i+1}. The graph G_i is called the invaded region at time i, and the graph $G = \cup_{i=0}^{\infty} G_i$ is called the *invasion percolation cluster* (IPC).

The first rigorous study of invasion percolation was done in '85 by Chayes-Chayes-Newman [14], who took a dynamical perspective: their questions were related to the evolution of the graph G_n as n increases. In the '90s and '00s, results focused on a more static perspective: properties of the full invaded region. For example, the fractal dimension of G was determined [23] along with finer properties of G like relations to other critical models [24] and analysis of the pond and outlet structure.

3.1.2 Acceptance profile

In this paper, we return to the earlier dynamical perspective and study the "acceptance profile" of the invasion. Roughly speaking, the acceptance profile $a_n(p)$ at value p and time n is the ratio

$$a_n(p) = \frac{\text{expected number of bonds invaded with weight in } [p, p + \mathrm{d}p]}{\text{expected number of bonds observed with weight in } [p, p + \mathrm{d}p]},$$

where both the numerator and denominator are computed until time n, and a bond is observed by time n if it is either invaded by time n or is on the boundary of the invasion at time n.

To get the rigorous definition of the acceptance profile along with the results of [14], we use the notations of [14]. Let $I_n \in \mathcal{E}^2$ be the invaded bond at time $n \geq 1$ and let x_n be the random weight of I_n (the weight $\omega(I_n)$). For any $y \in [0, 1]$, define $X_n(y)$ as the indicator that $x_n \leq y$:

$$X_n(y) = \begin{cases} 1 & \text{if } x_n \leq y \\ 0 & \text{otherwise} \end{cases}$$

Let R_n be the random number of new bonds which must be checked after the invasion of I_n (that is, $R_0 = 4$, $R_1 = 3$, and R_n is the number of boundary edges of G_n that were not boundary edges of G_{n-1}) and define $L_n := \sum_{j=0}^{n} R_j$ to be the total number of checked bonds until after the invasion of I_n. Clearly, $n \leq L_n \leq 4n$. Denote by v_n the value of the n^{th} checked bond. (Here we can enumerate the checked edges counted in R_n in any deterministic fashion.) Set $V_n(y)$ to be the indicator that $v_n \le y$:

$$V_n(y) = \begin{cases} 1 & \text{if } \; v_n \le y \\ 0 & \text{otherwise} \end{cases}$$

Then the acceptance profile at value x by time n is defined as

$$a_n(x) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}\left[\sum_{j=1}^{n} \Big(X_j(x+\epsilon) - X_j(x) \Big)\right]}{\mathbb{E}\left[\sum_{j=1}^{L_n} \Big(V_j(x+\epsilon) - V_j(x) \Big)\right]} \tag{3.1.1}$$

It is shown in [14, Proposition 4.1] that $a_n(x)$ is an analytic function of x.

An alternative representation for the acceptance profile will be useful for us. Let $\tilde{Q}_n(x) = \sum_{j=1}^{n} X_j(x)$ be the number of invaded edges until time n with weight $\le x$ and $\tilde{P}_n(x) = \sum_{j=1}^{L_n} V_j(x)$ be the number of checked edges until time n with weight $\le x$. From [14, Eq. (4.3)], one has

$$\mathbb{E}[\tilde{P}_n(x)] = x\mathbb{E}[L_n],$$

and so we can rewrite (3.1.1) as

$$a_n(x) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}[\tilde{Q}_n(x+\epsilon) - \tilde{Q}_n(x)]}{\epsilon \mathbb{E}[L_n]}. \tag{3.1.2}$$

Analysis of the IPC and the acceptance profile heavily involves tools from Bernoulli percolation, whose definition depends on a parameter $p \in [0,1]$. We will couple the percolation model to the IPC as follows. For every $e \in \mathcal{E}^2$ and any $p \in [0,1]$, we say that e is p-open if $\omega(e) \le p$; otherwise, we say that e is p-closed. Note that the variables $(\mathbf{1}_{\{e \text{ is } p\text{-open}\}})_{e \in \mathcal{E}^2}$ are i.i.d. Bernoulli random variables with parameter p. The main object of study in percolation is the connectivity properties of the graph whose edges consist of the p-open edges. If p is large, we expect this graph to contain very large (even infinite) components and if p is small we expect it to contain only small components. To formulate these ideas precisely, we say that a path (a finite or infinite sequence of edges $e_1, e_2, \ldots$ such that e_i and e_{i+1} share at least one endpoint) is p-open if all its edges are p-open, and we write $A \overset{p}{\longleftrightarrow} B$ for two sets of vertices A and B if there is a p-open path starting at a vertex in A and ending at a vertex in B. We also write $u \overset{p}{\longleftrightarrow} v$ for vertices u, v when $A = \{u\}$ and $V = \{v\}$, and we use the term "p-open cluster of u" for the set of vertices v such that $u \overset{p}{\longleftrightarrow} v$. Last, we write $u \overset{p}{\longleftrightarrow} \infty$ to mean that the p-open cluster of u is infinite. Given this setup, we define the critical threshold for percolation as

$$p_c = \sup\{p \in [0,1] : \theta(p) = 0\},$$

where

$$\theta(p) = \mathbb{P}(0 \overset{p}{\longleftrightarrow} \infty).$$

It is known that for all dimensions $d \geq 2$, one has $p_c \in (0,1)$, and for $d = 2$, $p_c = 1/2$. These facts and more can be seen in the standard reference [8].

In addition to p_c, there are other critical values that have been used in the past, and these have all been shown to be equal to p_c. The two that were used in [14] are

$$\pi_c = \sup\{p \in [0,1] : \mathbb{E}_p \#\{v : v \text{ is in the } p\text{-open cluster of } 0\} < \infty)\}, \text{ and}$$
$$\bar{p}_c = \sup\{p \in [0,1] : \mathbb{P}_p(\exists \text{ infinite } p\text{-open path in a half-space}) = 0\}.$$

In this language, and for general dimensions, the theorems of Chayes-Chayes-Newman state that

$$\lim_{n\to\infty} a_n(p) = \begin{cases} 1 & \text{if } p < \pi_c \\ 0 & \text{if } p > \bar{p}_c \end{cases} \tag{3.1.3}$$

Because π_c and $\bar{p}_c$ are both known to be equal to p_c (see [25, 26, 12]), this result specifies the limiting behavior of the acceptance profile at all values of $p \neq p_c$. Our main result, Theorem 3.3.1, shows that in two dimensions, the limiting behavior of $a_n(p_c)$ is different than that of $a_n(p)$ for any other value of p: it remains bounded away from zero and one.

3.1.3 Notation and outline

First we gather some notation used in the paper. For $n \geq 1$ let $B(n) = [-n, n]^2$ be the box of sidelength $2n$, and for $m < n$, let $\mathrm{Ann}(m, n)$ be the annulus $B(n) \setminus B(m)$. We will be interested in connection probabilities from points to boxes, so we set

$$\pi(p, n) = \mathbb{P}(0 \xleftrightarrow{p} \partial B(n) \text{ and } \pi(n) = \pi(p_c, n).$$

Many connection probabilities (or their complements) can be expressed in terms of connections on the dual graph $(\mathbb{Z}^2)^*$. To define it, let $(\mathbb{Z}^2)^* = \left(\frac{1}{2}, \frac{1}{2}\right) + \mathbb{Z}^2$ be the set of dual vertices and let $(\mathcal{E}^2)^*$ be the edges between nearest-neighbor dual vertices. For $x \in \mathbb{Z}^2$ we write $x^* = x + \left(\frac{1}{2}, \frac{1}{2}\right)$ for its dual vertex. For an edge $e \in \mathcal{E}^2$, we denote its endpoints (left, respectively right or bottom, respectively top) by $e_x, e_y \in \mathbb{Z}^2$. The edge $e^* = \left\{e_x + \left(\frac{1}{2}, \frac{1}{2}\right), e_y - \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ is called the edge dual to e. (It is the unique dual edge that bisects e.) A dual edge e^* is called p-open if e is p-open, and is p-closed otherwise. A dual path is a finite or infinite sequence of dual edges such that consecutive edges share at least one endpoint. A circuit (or dual circuit) is a finite path (or dual path) which has the same initial and final vertices. Last, in this paper, $\| \cdot \|_1$ means the L^1-norm.

In the next section, we give the proof of Theorem 3.3.1. It is split into three subsections. In Section 3.2, we introduce correlation length and useful results which are frequently used in two dimensional percolation. In Section 3.4, we prove the lower bound of Theorem 3.3.1 and in Section 3.5, we prove the upper bound of Theorem 3.3.1.

3.2 Preliminaries

We first introduce the finite-size scaling correlation length (see a more detailed survey in [11]). Let

$$\sigma(n,m,p) = \mathbb{P}(\exists \text{ a } p\text{-open horizontal crossing of } [0,n]\times[0,m]).$$

Here, a horizontal crossing is a path which remains in $[0,n]\times[0,m]$, with initial vertex in $\{0\}\times[0,m]$ and final vertex in $\{n\}\times\{0,m\}$. For any $\epsilon > 0$, we set

$$L(p,\epsilon) := \begin{cases} \min\{n : \sigma(n,n,p) \le \epsilon\} & \text{if } \ p < p_c \\ \min\{n : \sigma(n,n,p) \ge 1-\epsilon\} & \text{if } \ p > p_c \end{cases}$$

$L(p,\epsilon)$ is called the finite-size scaling correlation length and its scaling as $p \to p_c$ does not depend on ϵ, so long as ϵ is small enough. That is, there exists an $\epsilon_0 > 0$ such that for $\epsilon_1, \epsilon_2 \in (0, \epsilon_0]$, $L(p,\epsilon_1) \asymp L(p,\epsilon_2)$ as $p \to p_c$ [27, Eq. (1.24)]. For this reason, we set

$$L(p) = L(p,\epsilon_0).$$

Because $L(p) \to \infty$ as $p \to p_c$ [11, Prop. 4] and $L(p) \to 0$ as $p \to 0$ or $p \to 1$, the approximate inverses

$$p_n = \sup\{p > p_c : L(p) > n\}$$
$$q_n = \inf\{q < p_c : L(q) > n\}$$

are well-defined.

Next we list relevant and now standard properties of the correlation length with references to their proofs.

1. [27, Thm. 1] For $n \le L(p)$ and $p \neq p_c$,

$$\pi(p,n) \asymp \pi(n). \tag{3.2.1}$$

2. [27, Thm. 2] There are positive constants C_1 and C_2 such that for all $p > p_c$

$$\pi(L(p)) \leq \pi(p, L(p)) \leq C_1\theta(p) \leq C_1\pi(p, L(p)) \leq C_2\pi(L(p)). \tag{3.2.2}$$

3. [24, Eq. (2.8)] There are positive constants C_3, C_4 such that

$$\sigma(2mL(p), mL(p), p) \geq 1 - C_3\exp(-C_4 m), \quad \text{for} \;\; m > 1. \tag{3.2.3}$$

4. [24, Eq. (2.10)] There is a constant D such that

$$\lim_{\delta\downarrow 0} \frac{L(p-\delta)}{L(p)} \leq D \;\; \text{for} \;\; p > p_c. \tag{3.2.4}$$

5. [28, Cor. 3.15] There exists a constant $D_1 > 0$ such that

$$\frac{\pi(m)}{\pi(n)} \geq D_1\sqrt{\frac{n}{m}} \;\; \text{for}\; m \geq n \geq 1. \tag{3.2.5}$$

6. [11, Prop. 34] (Arm events). Fix $e = \{e_x, e_y\}$ and let $A_n^{2,2}$ be the event that e_x and e_y are connected to $\partial B(n)$ by p_c-open paths not containing e, and ${e_x}^*$ and ${e_y}^*$ are connected to $\partial B(n)^*$ by p_c-closed dual paths not containing e^*. Note that these four paths are disjoint and alternate. For $n \geq 1$,

$$\begin{aligned} (p_n - p_c)n^2\mathbb{P}(A_n^{2,2}) &\asymp 1 \\ (p_c - q_n)n^2\mathbb{P}(A_n^{2,2}) &\asymp 1. \end{aligned} \tag{3.2.6}$$

7. [11, Sec. 3.2] (Russo-Seymour-Welsh: RSW) For every $k, l \geq 1$, there exists $\delta_{k,l} > 0$

such that for all $p \in [p_c, p_n]$ (respectively $q \in [q_n, p_c]$),

$\mathbb{P}(\exists$ a p-open (respectively q-open) horizontal crossing of $[0, kn] \times [0, ln]) > \delta_{k,l}$

$\mathbb{P}(\exists$ a p-closed (respectively q-closed) horizontal dual crossing of $([0, kn] \times [0, ln])^* > \delta_{k,l}$.

In addition, applying the FKG inequality [8, Ch. 2], for all $p \in [p_c, p_n]$ (resp. $q \in [q_n, p_c]$),

$\mathbb{P}\big(\text{Ann}(n, kn)$ contains a p-open (resp. q-open) circuit around the origin$\big) > (\delta_{k,k-2})^4$

$\mathbb{P}\big(\text{Ann}(n, kn)^*$ contains a p-closed (resp. q-closed) dual circuit around the origin$\big) > (\delta_{k,k-2})^4$.

8. [24, 23] Let $|\mathcal{S}_n|$ be the number of invaded edges (edges in G) inside $B(n)$. Then,

$$\mathbb{E}|\mathcal{S}_n| \asymp n^2 \pi(n). \tag{3.2.7}$$

Last, we prove some lemmas that will be helpful in the proof of the main theorem. These lemmas will bound the random variables

$$R_n := \min\{k : I_i \subset B(k) \text{ for } i = 1, 2, \cdots, n\}$$
$$r_n := \max\{k : I_i \subset B(k)^c \text{ for all } i > n\}.$$

R_n is a radius of the invaded region at time n, and r_n is the largest size of box such that the invasion does not change in this box after time n.

Lemma 3.2.1. *There exists a constant $\mathcal{C}_1 > 0$ such that for all $n \geq 1$ and $C > 0$,*

$$\mathbb{P}(R_{\lfloor Cn^2\pi(n) \rfloor} < n) \leq \frac{\mathcal{C}_1}{C}.$$

Proof. The event $\{R_{\lfloor Cn^2\pi(n) \rfloor} < n\}$ implies that $|\mathcal{S}_n| \geq \lfloor Cn^2 \pi(n) \rfloor$. By Markov's inequality and (3.2.7),

$$\mathbb{P}(R_{\lfloor Cn^2\pi(n)\rfloor} < n) \le \mathbb{P}(|\mathcal{S}_n| \ge \lfloor Cn^2\pi(n)\rfloor) \le \frac{\mathbb{E}|\mathcal{S}_n|}{\lfloor Cn^2\pi(n)\rfloor} \le \frac{\mathcal{C}_1}{C}.$$

$\square$

Lemma 3.2.2. *For any $\eta_0 > 0$, there exists $\mathcal{C}_2 > 0$ such that for any $C \ge \mathcal{C}_2$ and $n \ge 1$,*

$$\mathbb{P}(r_{\lfloor Cn^2\pi(n)\rfloor} < 2n) \le \eta_0$$

Proof. For $k, m \ge 1$, we consider the event $D_{k,m}$ defined by the following conditions:

(i) There is a p_c-open circuit around the origin in $\mathrm{Ann}(2^{k+1}, 2^{k+1+\frac{m}{8}})$.

(ii) There is a $p_{2^{k+1+\frac{m}{4}}}$-closed dual circuit around the origin in $\mathrm{Ann}(2^{k+1+\frac{m}{8}}, 2^{k+1+\frac{m}{4}})^*$.

(iii) There is a p_c-open circuit around the origin in $\mathrm{Ann}(2^{k+1+\frac{m}{2}}, 2^{k+1+m})$.

(iv) The circuit from (iii) is connected to infinity by a $p_{2^{k+1+\frac{m}{4}}}$-open path.

(See Figure 3.1 for an illustration of $D_{k,m}$.)

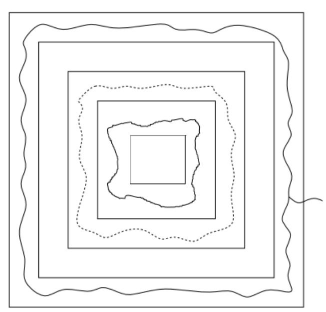

Figure 3.1: Illustration of the event $D_{k,m}$. The boxes, in order from smallest to largest, are $B(2^{k+1})$, $B(2^{k+1+\frac{m}{8}})$, $B(2^{k+1+\frac{m}{4}})$, $B(2^{k+1+\frac{m}{2}})$ and $B(2^{k+1+m})$. The solid circuit is p_c-open, the path to infinity is $p_{2^{k+1+\frac{m}{4}}}$-open, and the dotted path is $p_{2^{k+1+\frac{m}{4}}}$-closed.

For $j, k, m \ge 1$, we claim that

$$\big(\{R_j \ge 2^{k+1+m}\} \cap D_{k,m}\big) \subset \{r_j \ge 2^{k+1}\}. \tag{3.2.8}$$

To see why, suppose the left side occurs, and choose $\mathfrak{C}_1$ as a circuit from (i) in the definition of $D_{k,m}$, $\mathfrak{C}_2$ as a circuit from (ii), and $\mathfrak{C}_3$ as a circuit from (iii). Let n_1 be the time at which the invasion invades all of $\mathfrak{C}_1$ and for $i = 2, 3$, let n_i be the first time that the invasion invades an edge from $\mathfrak{C}_i$. Note that $n_1 \leq n_2 \leq n_3 \leq j$. (The last inequality holds because $R_j \geq 2^{k+1+m}$.)

After time n_3, the invasion has an unending supply of edges with weight $< p_{2^{k+1+\frac{m}{4}}}$ to invade, so it will never again take an edge with weight larger than that. Furthermore, at time n_2, the invasion must take an edge with weight larger than $p_{2^{k+1+\frac{m}{4}}}$. This implies that at some time $n_4 \in [n_2, n_3)$, the invasion invades an outlet: an edge $\hat{e}$ such that all edges invaded after time n_4 have weight $< \omega(\hat{e})$. Furthermore, this outlet can be chosen to have weight $\omega(\hat{e}) > p_{2^{k+1+\frac{m}{4}}} > p_c$.

Directly before time n_4, the entire boundary of the invasion (excluding $\hat{e}$ itself) consists of edges with weight $> \omega(\hat{e})$. Since invaded weights beyond time n_4 are $< \omega(\hat{e})$, none of these boundary edges will ever be invaded. Therefore all invaded edges after time n_4 are invaded through $\hat{e}$. In other words, if e is any edge invaded after time n_4, there is a path $P(e)$ connecting $\hat{e}$ to e consisting of edges with weight $< \omega(\hat{e})$ and which are invaded after time n_4. It is important to note that $P(e)$ cannot touch $\mathfrak{C}_1$. Indeed, if were to contain an edge f which shared an endpoint with an edge on $\mathfrak{C}_1$ (including the possibility that $f \in \mathfrak{C}_1$), then f would be accessible to the invasion at time n_1, and so f would be invaded before time n_4, a contradiction.

Finally, to prove (3.2.8), assume that $r_j < 2^{k+1}$. Then there is some time $j' > j$ at which the invasion invades an edge e in $B(2^{k+1})$. Since $j' > n_4$, there is a path $P(e)$ from $\hat{e}$ to e as in the preceding paragraph which cannot touch $\mathfrak{C}_1$. This means $\hat{e}$ is in the interior of $\mathfrak{C}_1$. On the other hand, if f is any edge of $\mathfrak{C}_3$ (necessarily invaded after time n_4), the path $P(f)$ connecting $\hat{e}$ to f would then toucn $\mathfrak{C}_1$, a contradiction. This shows (3.2.8).

Applying (3.2.8) for $C > 0$ and $k, m \geq 1$, we obtain

$$\mathbb{P}(r_{\lfloor C2^{2k}\pi(2^k)\rfloor} < 2^{k+1}) \leq 2\max\{\mathbb{P}(R_{\lfloor C2^{2k}\pi(2^k)\rfloor} < 2^{k+1+m}), \mathbb{P}(D^c_{k,m}).\} \tag{3.2.9}$$

As in [13, proof of Thm. 5], the RSW theorem implies that $\mathbb{P}(D^c_{k,m}) \leq e^{-\delta m}$ for some $\delta > 0$ uniformly in k, so we can fix m so that

$$\mathbb{P}(D^c_{k,m}) \leq \frac{\eta_0}{2} \text{ for all } k \geq 1. \tag{3.2.10}$$

From Lemma 3.2.1 and the fact that $\pi(n)$ is decreasing in n, for any $C \geq (2\mathcal{C}_1 2^{2+2m})/\eta_0 =: \mathcal{C}_2$, we get

$$\mathbb{P}\big(R_{\lfloor C2^{2k}\pi(2^k)\rfloor} < 2^{k+1+m}\big) \leq \mathbb{P}\big(R_{\lfloor (2\mathcal{C}_1/\eta_0)2^{2(k+1+m)}\pi(2^{k+1+m})\rfloor} < 2^{k+1+m}\big) \leq \frac{\eta_0}{2}.$$

Combining this with (3.2.9) and (3.2.10), we find that for $C \geq \mathcal{C}_2$,

$$\mathbb{P}\big(r_{\lfloor C2^{2k}\pi(2^k)\rfloor} < 2^{k+1}\big) \leq \eta_0,$$

and this completes the proof for n of the form 2^k.

For general n, we let $k = k(n) := \lfloor \log_2 n \rfloor$, so that for any $C \geq 4\mathcal{C}_2$,

$$\mathbb{P}(r_{\lfloor Cn^2\pi(n)\rfloor} < 2n) \leq \mathbb{P}(r_{\lfloor \mathcal{C}_2 2^{2(k+1)}\pi(2^{k+1})\rfloor} < 2^{k+2}) \leq \eta_0.$$

□

3.3 Main result

Theorem 3.3.1. *In two dimensions, where $p_c = 1/2$,*

$$0 < \liminf_{n\to\infty} a_n(p_c) \leq \limsup_{n\to\infty} a_n(p_c) < 1.$$

3.4 Lower bound

In this section, we show that

$$\liminf_{n\to\infty} a_n(p_c) > 0. \tag{3.4.1}$$

The first step is to show that it suffices to prove this result for only a certain subsequence of values of n. Namely, we first prove that if there exists $\mathcal{C}_3 > 0$ such that

$$\liminf_{n\to\infty} a_{\lfloor \mathcal{C}_3 n^2 \pi(n)\rfloor}(p_c) > 0, \tag{3.4.2}$$

then (3.4.1) follows.

So assume that (3.4.2) holds, and let

$$k = k(n) := \max\{\ell : \mathcal{C}_3 \ell^2 \pi(\ell) \le n\}.$$

(Note that this k actually exists for large n since $\pi(\ell) \ge D_1/\sqrt{\ell}$ by (3.2.5).) Since $\tilde{Q}_n(p_c + \epsilon) - \tilde{Q}_n(p_c)$ is increasing in n,

$$\tilde{Q}_n(p_c+\epsilon) - \tilde{Q}_n(p_c) \ge \tilde{Q}_{\lfloor \mathcal{C}_3 k^2\pi(k)\rfloor}(p_c+\epsilon) - \tilde{Q}_{\lfloor \mathcal{C}_3 k^2\pi(k)\rfloor}(p_c).$$

So using $n \le L_n \le 4n$, we obtain

$$\begin{aligned} a_n(p_c) = \lim_{\epsilon\downarrow 0} \frac{\mathbb{E}[\tilde{Q}_n(p_c+\epsilon) - \tilde{Q}_n(p_c)]}{\epsilon \mathbb{E}[L_n]} &\ge \lim_{\epsilon\downarrow 0} \frac{\mathbb{E}[\tilde{Q}_{\lfloor \mathcal{C}_3 k^2\pi(k)\rfloor}(p_c+\epsilon) - \tilde{Q}_{\lfloor \mathcal{C}_3 k^2\pi(k)\rfloor}(p_c)]}{\epsilon \mathbb{E}[L_{\lfloor \mathcal{C}_3 k^2\pi(k)\rfloor}]} \frac{\mathbb{E}[L_{\lfloor \mathcal{C}_3 k^2\pi(k)\rfloor}]}{\mathbb{E}[L_n]} \\ &\ge a_{\lfloor \mathcal{C}_3 k^2\pi(k)\rfloor}(p_c) \frac{\mathcal{C}_3 k^2 \pi(k)}{8n} \end{aligned}$$

Thus to conclude (3.4.1) from (3.4.2), it suffices to show that $\liminf_{n\to\infty} k^2\pi(k)/n$ is positive. For large n, $k(n)$ is greater than 1; therefore,

$$\frac{k^2\pi(k)}{n} \ge \frac{k^2\pi(k)}{\mathcal{C}_3(k+1)^2\pi(k+1)} \ge \mathcal{C}_3^{-1}\left(\frac{k}{k+1}\right)^2 \ge \frac{1}{4\mathcal{C}_3} > 0.$$

To prove (3.4.2), we use the following lemma, which bounds the k^{th} moment of the number of edges of the IPC with $(p_c, p_c + \epsilon]$ in $B(n)$.

Lemma 3.4.1. *Let $\mathcal{Y}_n(\epsilon)$ be the number of invaded edges in $B(n)$ with $(p_c, p_c+\epsilon]$ for $\epsilon > 0$. There exist positive constants $\mathcal{C}_4$ and $\mathcal{C}_5 = \mathcal{C}_5(t)$ such that for all $n \geq 1$,*

$$\liminf_{\epsilon \downarrow 0} \frac{\mathbb{E}|\mathcal{Y}_n(\epsilon)|}{\epsilon} \geq \mathcal{C}_4 n^2 \pi(n)$$

and

$$\mathbb{E}|\mathcal{Y}_n(\epsilon)|^t \leq \mathcal{C}_5 \left(\epsilon n^2 \pi(n)\right)^t \text{ for all } t \geq 1 \text{ and } \epsilon > 0.$$

Assuming this lemma for the moment, we can derive (3.4.2). From Lemma 3.2.2, we can choose $\mathcal{C}_3$ so that

$$\mathbb{P}(r_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor} < 2n) \leq \frac{\mathcal{C}_4^2}{16\mathcal{C}_5(2)} \text{ for all } n \geq 1.$$

On the event $\{r_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor} \geq 2n\}$, the IPC in $B(2n)$ does not change after time $\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor$. It follows that the number of invaded edges with $(p_c, p_c+\epsilon]$ until time $\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor$ is at least $\mathcal{Y}_{2n}(\epsilon)$, which is the number of invaded edges with $(p_c, p_c + \epsilon]$ in $B(2n)$. By Lemma 3.2.2, Lemma 3.4.1 and the Cauchy-Schwarz inequality, if ϵ is sufficiently small,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=1}^{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor} \Big(X_j(p_c+\epsilon) - X_j(p_c)\Big)\right] \\
&\geq \mathbb{E}\left[\sum_{j=1}^{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor} \Big(X_j(p_c+\epsilon) - X_j(p_c)\Big) \cdot \mathbf{1}_{\{r_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor} \geq 2n\}}\right] \\
&\geq \mathbb{E}\left[\mathcal{Y}_{2n}(\epsilon) \cdot \mathbf{1}_{\{r_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor} \geq 2n\}}\right] \geq \frac{\mathcal{C}_4}{2} \epsilon (2n)^2 \pi(2n) - \mathbb{E}\left[\mathcal{Y}_{2n}(\epsilon) \cdot \mathbf{1}_{\{r_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor} < 2n\}}\right] \\
&\geq \frac{\mathcal{C}_4}{2} \epsilon (2n)^2 \pi(2n) - \sqrt{\mathcal{C}_5(2) \left(\epsilon (2n)^2 \pi(2n)\right)^2 \frac{\mathcal{C}_4^2}{16\mathcal{C}_5(2)}} \\
&\geq \frac{\mathcal{C}_4 \epsilon (2n)^2 \pi(2n)}{4}.
\end{aligned}$$

Combining this with (3.1.2), (3.2.5), and the fact that $n \le \mathbb{E}[L_n] \le 4n$, we obtain

$$a_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor}(p_c) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}[\tilde{Q}_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor}(p_c+\epsilon) - \tilde{Q}_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor}(p_c)]}{\epsilon \mathbb{E}[L_{\lfloor \mathcal{C}_3 n^2 \pi(n) \rfloor}]} \ge \lim_{\epsilon \downarrow 0} \frac{\mathcal{C}_4 \epsilon (2n)^2 \pi(2n)/4}{4\epsilon \mathcal{C}_3 n^2 \pi(n)}$$
$$= \frac{\mathcal{C}_4 D_1}{4\mathcal{C}_3 \sqrt{2}},$$

which is positive uniformly in n. This shows (3.4.2).

The last step is to prove Lemma 3.4.1.

Proof of Lemma 3.4.1. The proof of the upper bound is similar to that of Járai [24, Theorem 1], which shows an upper bound for $|\mathcal{S}_n|$ (that result does not involve a condition on the weight $\omega(e)$) so we will omit some details. We will follow that proof, but make the events independent of $\omega(e)$ so that we can insert the condition $\omega(e) \in (p_c, p_c + \epsilon]$.

We will restrict to n of the form 2^K, as the general result follows from this and monotonicity of π_n. Let A_k be Ann$(2^k, 2^{k+1})$, and $\mathcal{Y}_{A_k}$ be the number of IPC edges in Ann$(2^k, 2^{k+1})$ with the weight in $(p_c, p_c + \epsilon]$. Then, $B(n) = \cup_{k=1}^K A_k$ and $\mathcal{Y}_n(\epsilon) = \sum_{k=1}^K \mathcal{Y}_{A_k}$. Define a sequence $p_k(0) > p_k(1) > \cdots > p_c$ as follows. Let $\log^{(0)} k = k$, and let $\log^{(j)} k = \log(\log^{(j-1)} k)$ for $j \ge 1$ if the right-hand side is defined. For $k > 10$, we define

$$\log^* k = \min\{j > 0 : \log^{(j)} k \text{ is defined and } \log^{(j)} k \le 10\}.$$

Then $\log^{(j)} k > 2$, for $j = 0, 1, \cdots, \log^* k$ and $k > 10$. Let

$$p_k(j) = \inf\left\{p > p_c : L(p) \le \frac{2^k}{C_5 \log^{(j)} k}\right\}, \quad j = 0, 1, \cdots, \log^* k,$$

where the constant C_5 will be chosen later. With (3.2.4) and [24, Eq. (2.15)], we get

$$C_5 \log^{(j)} k \le \frac{2^k}{L(p_k(j))} \le DC_5 \log^{(j)} k \tag{3.4.3}$$

For any fixed $e \subset A_k$ we define

$$H_k(j) = \{\exists \text{ a } p_k(j)\text{-open circuit } \mathcal{D} \text{ around the origin in } A_{k-1} \text{ and } \mathcal{D} \xleftrightarrow{p_k(j)} \infty\}$$
$$H_k^e(j) = \{H_k(j) \text{ occurs and } \mathcal{D} \xleftrightarrow{p_k(j)} \infty \text{ without using the edge } e\}. \tag{3.4.4}$$

To give a lower bound for the probability of $H_k(j)$, Járai constructed an infinite $p_k(j)$-open path starting from $\partial B(2^k)$ using standard 2D constructions only to the right of $B(2^k)$. (See [24, Fig 1]). Similarly, to lower bound the probability of $H_k(j)^e$, we build, in addition to Járai's path, an infinite $p_k(j)$-open path starting from $\partial B(2^k)$ in the left of $B(2^k)$. The existence of such disjoint two infinite $p_k(j)$-open paths imply the event $\{\mathcal{D} \xleftrightarrow{p_k(j)} \infty$ without using $e\}$ for any fixed edge $e \in A_k$. As in [24, Eq. (2.17)], we obtain

$$J_k(j) \cap \left(\bigcap_{m=0}^{\infty} J_{k,L}^m(j)\right) \cap \left(\bigcap_{m=0}^{\infty} J_{k,R}^m(j)\right) \subseteq H_k^e(j) \tag{3.4.5}$$

where for $m \geq 0$,

$$J_k = \{\exists \text{ a } p_k(j)\text{-open circuit in } A_{k-1}\}$$
$$J_{k,R}^m = J_{k,R}^{m,h} \cap J_{k,R}^{m,v}, \text{ and } J_{k,L}^m = J_{k,L}^{m,h} \cap J_{k,L}^{m,v}$$
$$J_{k,R}^{m,h} = \{\exists \text{ a } p_k(j)\text{-open horizontal crossing of } [2^{k-2+m}, 2^{k+m}] \times [-2^{k-2+m}, 2^{k-2+m}]\}$$
$$J_{k,L}^{m,h} = \{\exists \text{ a } p_k(j)\text{-open horizontal crossing of } [-2^{k+m}, -2^{k-2+m}] \times [-2^{k-2+m}, 2^{k-2+m}]\}$$
$$J_{k,R}^{m,v} = \{\exists \text{ a } p_k(j)\text{-open vertical crossing of } [2^{k-1+m}, 2^{k+m}] \times [-2^{k-1+m}, 2^{k-1+m}]\}$$
$$J_{k,L}^{m,v} = \{\exists \text{ a } p_k(j)\text{-open vertical crossing of } [-2^{k+m}, -2^{k-1+m}] \times [-2^{k-1+m}, 2^{k-1+m}]\}.$$

By (3.2.3) and (3.4.3), (See [24, Eqs. (2.19), (2.20)]),

$$\mathbb{P}(J_k(j)^c) \leq 16C_3 \exp\left\{-\frac{1}{4}C_4C_5 \log^{(j)} k\right\} \quad \text{and}$$
$$\mathbb{P}(J_{k,R}^m(j)^c \cup J_{k,L}^m(j)^c) \leq 4C_3 \exp\left\{-\frac{1}{2}C_4C_5 2^m \log^{(j)} k\right\}.$$

By these inequalities, one gets

$$\begin{aligned}\mathbb{P}(H_k^e(j)^c) &\leq \mathbb{P}\left(J_k(j)\right)^c + \sum_{m=0}^{\infty} \mathbb{P}\left((J_{k,R}^m(j)^c \cup J_{k,L}^m(j)^c\right) \\ &\leq (16C_3 + C_6)\exp\left\{-\frac{1}{4}C_4C_5\log^{(j)} k\right\}.\end{aligned}$$

We write C_7 as $16C_3 + C_6$ and c_1 as $\frac{C_4C_5}{4}$ for short. Then,

$$\mathbb{P}(H_k^e(j)^c) \leq C_7 \exp\{-c_1 \log^{(j)} k\}. \tag{3.4.6}$$

The constant c_1 can be made large by choosing C_5 large.

To estimate the mean of $\mathcal{Y}_{A_k}$, we decompose

$$\mathbb{E}\mathcal{Y}_{A_k} = \mathbb{E}\left[\mathcal{Y}_{A_k}; H_k(0)^c\right] + \left(\sum_{j=1}^{\log^* k} \mathbb{E}\left[\mathcal{Y}_{A_k}; H_k(j-1) \cap H_k(j)^c\right]\right) + \mathbb{E}\left[\mathcal{Y}_{A_k}; H_k(\log^* k)\right]. \tag{3.4.7}$$

By (3.4.6) and independence,

$$\begin{aligned}\mathbb{E}\left[\mathcal{Y}_{A_k}; H_k(0)^c\right] \leq \mathbb{E}\left[\mathcal{Y}_{A_k}; H_k^e(0)^c\right] &\leq \sum_{e \in A_k} \mathbb{P}(\omega(e) \in (p_c, p_c + \epsilon], H_k^e(0)^c) \\ &= |A_k| \mathbb{P}(\omega(e) \in (p_c, p_c + \epsilon]) \mathbb{P}(H_k^e(0)^c) \\ &\leq |A_k| \epsilon C_8 e^{-c_1 k}.\end{aligned} \tag{3.4.8}$$

Next, since $\omega(e)$ is independent of $H_k^e(j) \cap \{e \xleftrightarrow{p_k(j)} \infty\}$,

$$\begin{aligned}\mathbb{E}|\mathcal{Y}_{A_k}; H_k(j-1) \cap H_k(j)^c| &= \sum_{e \in A_k} \mathbb{P}(\omega(e) \in (p_c, p_c + \epsilon] \cap \{e \xleftrightarrow{p_k(j-1)} \infty\} \cap H_k^e(j)^c) \\ &= \epsilon \sum_{e \in A_k} \mathbb{P}(e \xleftrightarrow{p_k(j-1)} \infty, H_k^e(j)^c).\end{aligned}$$

Applying the FKG inequality and (3.4.6) to this, we obtain

$$\mathbb{E}[\mathcal{Y}_{A_k}; H_k(j-1) \cap H_k(j)^c] \leq |A_k|\epsilon\theta(p_k(j-1))\; C_7 \exp\{-c_1 \log^{(j)} k\}. \tag{3.4.9}$$

The third term of (3.4.7) is bounded above by

$$|A_k|\epsilon\theta(p_k(\log^* k)). \tag{3.4.10}$$

Using (3.2.2), (3.2.5) and (3.4.3),

$$\theta(p_k(j)) \leq \frac{\pi(2^k)}{D_1}(DC_5 \log^{(j)} k)^{1/2}.$$

Applying this inequality after placing (3.4.8), (3.4.9), and (3.4.10) into (3.4.7), we obtain

$$\mathbb{E}\mathcal{Y}_{A_k} \leq C_9|A_k|\epsilon\pi(2^k)\left[\frac{\exp\{-c_1 k\}}{\pi(2^k)} + \left\{\sum_{j=1}^{\log^* k}(\log^{j-1} k)^{1/2-c_1}\right\} + 1\right].$$

Since $\pi(2^k) \geq C_{10}2^{-k/2}$ from (3.2.5), we can choose C_5 (and therefore c_1) so large that

$$\frac{\exp\{-c_1 k\}}{\pi(2^k)} + \left\{\sum_{j=1}^{\log^* k}(\log^{j-1} k)^{1/2-c_1}\right\} + 1 \quad \text{is bounded in } k,$$

and so $\mathbb{E}\mathcal{Y}_{A_k} \leq C_{11}\epsilon 2^{2k}\pi(2^k)$. Recalling $n = 2^K$, we obtain from this and (3.2.5) that

$$\begin{aligned}\mathbb{E}\mathcal{Y}_n(\epsilon) = \sum_{k=1}^{K}\mathbb{E}\mathcal{Y}_{A_k} \leq C_{11}\epsilon 2^{2K}\pi(2^K)\sum_{k=1}^{K}\frac{2^{2k}\pi(2^k)}{2^{2K}\pi(2^K)} &\leq \frac{C_{11}}{D_1}\epsilon 2^{2K}\pi(2^K)\sum_{k=1}^{K}2^{2(k-K)}2^{-\frac{1}{2}(k-K)} \\ &\leq C_{12}\epsilon n^2\pi(n),\end{aligned}$$

completing the proof of the upper bound when $t = 1$. The extension to larger t uses the same ideas as in [24] and [29, Sec. 3], so we omit it.

We now turn to the lower bound. For $k \geq 1$, $\epsilon > 0$, and any $e \subset A_k$, we let $L_k(e)$ be the event that the following hold:

(a) There exists a p_c-open circuit $\mathcal{D}$ around the origin in A_{k-2}.

(b) There exists a $(p_c + \epsilon)$-closed dual circuit around the origin in A_{k+2}.

(c) $\mathcal{D}$ is connected to the edge $e \in A_k$ by a p_c-open path in $B(2^k)$.

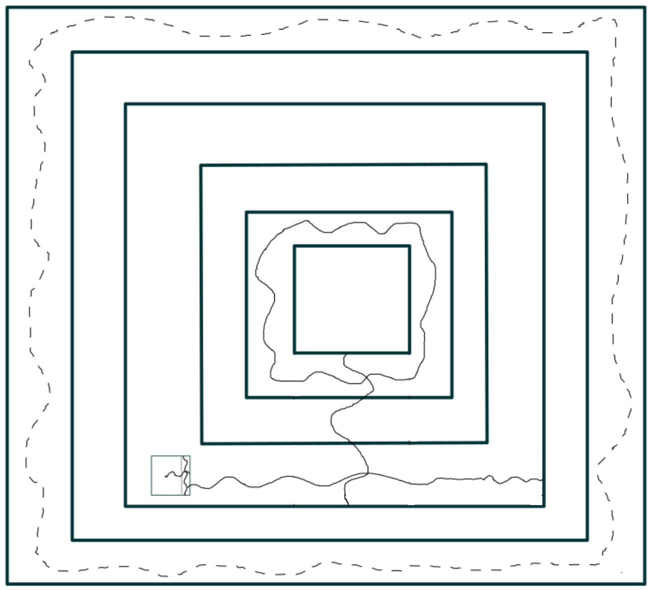

Figure 3.2: The event $L_k(e)$. The boxes, in order from smallest to largest, are $B(2^{k-2})$, $B(2^{k-1})$, $B(2^k)$, $B(2^{k+1})$, $B(2^{k+2})$ and $B(2^{k+3})$. The solid curves are p_c-open and the dotted curve is a $(p_c + \epsilon)$-closed dual circuit.

(See Figure 3.2 for an illustration of $L_k(e)$).

If the events described in (a) and (b) both occur, each $(p_c + \epsilon)$-open edge connected to $\mathcal{D}$ by a $(p_c + \epsilon)$-open path will eventually be invaded. Since the event in (b) depends on edge-variables for edges outside of $B(2^{k+1})$, (b) is independent of both (a) and (c). In addition, the events (a) and (c) are increasing. So, by the FKG inequality and the RSW theorem,

$$\mathbb{P}(L_k(e)) \geq \mathbb{P}((a)) \times \mathbb{P}((b)) \times \mathbb{P}((c)) \geq c_2 \mathbb{P}((b)) \times \mathbb{P}((c)).$$

By a gluing argument [8, Ch. 11] using the FKG inequality and the RSW theorem, $\mathbb{P}((c)) \geq c_3 \pi(2^k)$. Furthermore, as long as ϵ is so small that $p_c + \epsilon < p_{2^{k+2}}$, then the RSW theorem implies that $\mathbb{P}((b)) \geq c_4$. This means that for such ϵ, one has $\mathbb{P}(L_k(e)) \geq c_2c_3c_4\pi(2^k)$. Since $\omega(e)$ and the event $L_k(e)$ are independent,

$$\mathbb{E}\mathcal{Y}_{A_k} = \sum_{e\in A_k} \mathbb{P}(e \in \text{IPC}, \omega(e) \in (p_c, p_c+\epsilon]) \geq \sum_{e\in A_k} \mathbb{P}(L_k(e), \omega(e) \in (p_c, p_c+\epsilon])$$
$$\geq c_2c_3c_4\epsilon 2^{2k}\pi(2^k).$$

For a given $n \geq 1$, choose $k = \lfloor \log_2 n \rfloor$ to complete the proof:

$$\mathbb{E}\mathcal{Y}_n(\epsilon) \geq \mathbb{E}\mathcal{Y}_{A_k} \geq c_2c_3c_4\epsilon 2^{2k}\pi(2^k) \geq c_5\epsilon n^2\pi(n).$$

□

3.5 Upper bound

In this section, we show that

$$\limsup_{n\to\infty} a_n(p_c) < 1. \tag{3.5.1}$$

To prove (3.5.1), we define

$$\Xi_n(\epsilon) = \left[\tilde{P}_n(p_c+\epsilon) - \tilde{P}_n(p_c)\right] - \left[\tilde{Q}_n(p_c+\epsilon) - \tilde{Q}_n(p_c)\right],$$

as the number of edges with weight in the interval $(p_c, p_c+\epsilon]$ which the invasion observes until time n but does not invade, and we give the following proposition.

Proposition 3.5.1. *There exists $\mathcal{C}_6 > 0$ and a function G on $[0,\infty)$ with $\inf_{r\in[0,m]} G(r) > 0$ for each $m \geq 0$ such that for any $C \geq \mathcal{C}_6$, any $n \geq 1$, and any $\epsilon > 0$,*

$$\mathbb{E}\Xi_{\lfloor Cn^2\pi(n)\rfloor}(\epsilon) \geq G(C)\epsilon n^2\pi(n).$$

Assuming Proposition 3.5.1 for the moment, let $C \geq \mathcal{C}_6$, and use $\mathbb{E}L_n \leq 4n$ for

$$a_{\lfloor Cn^2\pi(n)\rfloor}(p_c) = \lim_{\epsilon\downarrow 0} \frac{\mathbb{E}\left[\tilde{Q}_{\lfloor Cn^2\pi(n)\rfloor}(p_c+\epsilon) - \tilde{Q}_{\lfloor Cn^2\pi(n)\rfloor}(p_c)\right]}{\mathbb{E}\left[\tilde{P}_{\lfloor Cn^2\pi(n)\rfloor}(p_c+\epsilon) - \tilde{P}_{\lfloor Cn^2\pi(n)\rfloor}(p_c)\right]} = \lim_{\epsilon\downarrow 0}\left(1 - \frac{\mathbb{E}\Xi_{\lfloor Cn^2\pi(n)\rfloor}(\epsilon)}{\mathbb{E}L_{\lfloor Cn^2\pi(n)\rfloor}}\right)$$
$$\leq \lim_{\epsilon\downarrow 0}\left(1 - \frac{G(C)\epsilon n^2\pi(n)}{4C\epsilon n^2\pi(n)}\right)$$
$$= 1 - \frac{G(C)}{4C}. \qquad (3.5.2)$$

Now note that any $n \geq \mathcal{C}_6$ can be written in the form $\lfloor Ch^2\pi(h)\rfloor$ for some integer $h \geq 1$ and some $C \in [\mathcal{C}_6, 4\mathcal{C}_6]$. To see why, observe that any $n \geq \mathcal{C}_6$ is in some interval of the form $[\mathcal{C}_6 h^2\pi(h), \mathcal{C}_6(h+1)^2\pi(h+1))$ for some $h \geq 1$ (since $h^2\pi(h) \to \infty$ as $h \to \infty$ by (3.2.5)). Then because

$$\frac{\mathcal{C}_6(h+1)^2\pi(h+1)}{\mathcal{C}_6 h^2\pi(h)} = \left(1+\frac{1}{h}\right)^2 \frac{\pi(h+1)}{\pi(h)} \leq 4,$$

we see that $n = \lfloor C_* \mathcal{C}_6 h^2\pi(h)\rfloor$ for some $C_* \in [1,4]$. By (3.5.2), then, we obtain

$$a_n(p_c) \leq 1 - \frac{\inf_{r\in[\mathcal{C}_6, 4\mathcal{C}_6]} G(r)}{4\mathcal{C}_6},$$

and this implies (3.5.1).

In the remainder of this section, we prove Proposition 3.5.1.

Proof of Proposition 3.5.1. For notational convenience, let $t_n = \lfloor Cn^2\pi(n)\rfloor$. To prove a lower bound on $\Xi_{t_n}(\epsilon)$, we will construct a large p_c-open cluster such that with positive probability, independent of n, the invasion has intersected this cluster at time t_n and has explored a positive fraction of its boundary edges, but has not yet absorbed the entire cluster. These explored boundary edges will have probability of order ϵ to have weight in the interval $(p_c, p_c+\epsilon]$, so our lower bound on $\mathbb{E}\Xi_{t_n}(\epsilon)$ will be of order ϵ times the size of this explored boundary, which will itself be of order $n^2\pi(n)$.

To construct this cluster, we need several definitions.

Definition 3.5.2. *Define the event $D(n)$ that the following conditions hold:*

1. *There exists a q_n-open circuit around the origin in Ann($n, 2n$).*

2. *There exists an edge $f \in$ Ann$(6n, 7n)$ with $\omega(f) \in (q_n, p_c)$ such that:*

    (a) *there exists a p_c-closed dual path P around the origin in Ann($4n, 8n$)$^* \backslash \{f^*\}$ that is connected to the endpoints of f^* so that $P \cup \{f^*\}$ is a dual circuit around the origin, and*

    (b) *there exists a p_c-open path connecting an endpoint of f to $B(n)$, and another disjoint p_c-open path connecting the other endpoint of f to $\partial B(16n)$.*

3. *There exists a p_c-open circuit around the origin in Ann($8n, 16n$).*

For $e \subset Ann(2n, 4n)$, define $D^e(n)$ as the event that $D(n)$ occurs without using the edge e. (That is, $D(n)$ occurs and the first connection listed in 2(b) does not use e.)

See Figure 3.3 for an illustration of $D(n)$.

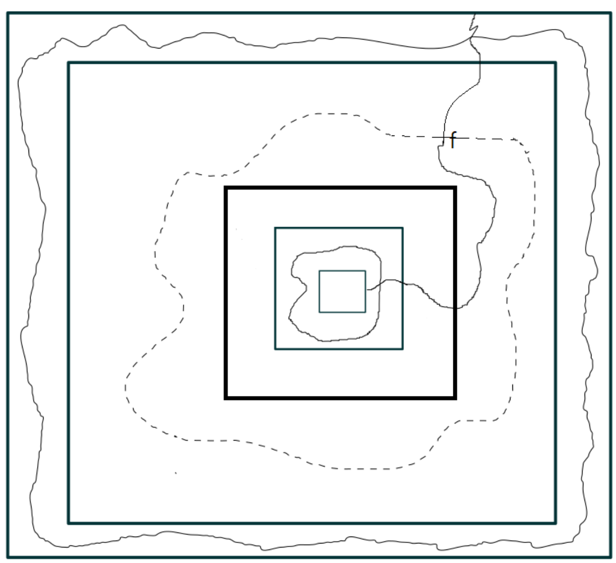

Figure 3.3: The event $D(n)$. The boxes, in order from smallest to largest, are $B(n)$, $B(2n)$, $B(4n)$, $B(8n)$ and $B(16n)$. The solid circuit in Ann$(n, 2n)$ is q_n-open and the path from $\partial B(n)$ to f is p_c-open; the dotted dual path in Ann$(4n, 8n)$ is p_c-closed, $\omega(f) \in (q_n, p_c)$, and the other solid paths are p_c-open.

When the event $D(n)$ occurs, we can define $\mathcal{C}_*$ as the innermost q_n-open circuit around the origin in $Ann(n, 2n)$ and D_* as the outermost p_c-open circuit around the origin in $Ann(8n, 16n)$. Note that on $D(n)$, the circuits $\mathcal{C}_*$ and D_* are part of the same p_c-open cluster; this will form part of our "large cluster" referenced above. We need to make sure that we have started to invade this cluster, but are not yet done at time t_n, so we define stopping times

$$
\begin{aligned}
t_{D_*} &= \text{ first time at which the invasion invades an edge from } D_* \\
T_{D_*} &= \text{ first time at which the invasion invades the entire } p_c\text{-open cluster of } D_*.
\end{aligned}
$$

Note that on $D(n)$, we have $t_{D_*} \leq T_{D_*}$ and trivially,

$$
\mathbb{E}\Xi_{t_n}(\epsilon) \geq \mathbb{E}\Xi_{t_n}(\epsilon)\mathbf{1}_{D(n)\cap\{t_{D_*}\leq t_n < T_{D_*}\}}. \tag{3.5.3}
$$

The next lemma shows that on the events listed on the right, $\Xi_{t_n}(\epsilon)$ is, on average, at least order ϵ times the cardinality of a certain subset of the edge boundary of the p_c-open cluster of D_*. For this we define the size Y_n of this subset:

$$
Y_n = \#\{e \subset Ann(2n, 4n) : \omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\}.
$$

Lemma 3.5.3. *For any* $n \geq 1$,

$$
\mathbb{E}\Xi_{t_n}(\epsilon)\mathbf{1}_{D(n)\cap\{t_{D_*}\leq t_n < T_{D_*}\}} \geq \frac{\epsilon}{1-p_c}\mathbb{E}Y_n\mathbf{1}_{D(n)\cap\{t_{D_*}\leq t_n < T_{D_*}\}}.
$$

Proof. First we let

$$
\hat{Y}_n = \#\{e \subset Ann(2n, 4n) : \omega(e) \in (p_c, p_c+\epsilon], e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\}.
$$

On the event $D(n) \cap \{t_{D_*} \leq t_n < T_{D_*}\}$, any edge in the set which defines $\hat{Y}_n$ will be observed by the invasion until time t_n but will not be invaded (that is, it is counted in the definition of $\Xi_n(\epsilon)$). To see why, let e be an edge in the set which defines $\hat{Y}_n$. First, we must show that e is not invaded at time t_n. This is because, in order for the invasion to even observe e, it must first pass through the circuit $\mathcal{C}_*$. Since $\omega(e) > p_c$, the invasion will invade the entire p_c-open cluster of $\mathcal{C}_*$ (which equals the p_c-open cluster of D_*) before it invades e. Since $t_n < T_{D_*}$, e cannot be invaded at time t_n. Second, we must show that e is observed by time t_n. The reason is that since $t_{D_*} \leq t_n$, at time t_n, the invasion has already invaded an edge from D_*. Since $D(n)$ occurs, the edge f must therefore be invaded before time $t_{D_*} \leq t_n$. Before f can be invaded, the entire q_n-open cluster of $\mathcal{C}_*$ must be invaded, so at least one endpoint of e is in the invasion at time t_n. This means that e is observed by time t_n. In conclusion,

$$\begin{aligned}
& \mathbb{E}\Xi_{t_n}(\epsilon)\mathbf{1}_{D(n)\cap\{t_{D_*}\leq t_n<T_{D_*}\}} \\
\geq\ & \mathbb{E}\hat{Y}_n\mathbf{1}_{D(n)\cap\{t_{D_*}\leq t_n<T_{D_*}\}} \\
=\ & \sum_{e\subset \text{Ann}(2n,4n)} \mathbb{P}(\omega(e)\in(p_c,p_c+\epsilon], e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D(n), t_{D_*}\leq t_n < T_{D_*}).
\end{aligned}$$

The second and final step is to show that for all $e \subset \text{Ann}(2n, 4n)$, we have

$$\begin{aligned}
& \mathbb{P}(\omega(e)\in(p_c,p_c+\epsilon], e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D(n), t_{D_*}\leq t_n < T_{D_*}) \\
=\ & \frac{\epsilon}{1-p_c}\mathbb{P}(\omega(e)>p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D(n), t_{D_*}\leq t_n < T_{D_*}).
\end{aligned} \tag{3.5.4}$$

Once this is done, we can sum the right side and obtain the statement of the lemma.

To argue for (3.5.4), we need to be able to decouple the value of $\omega(e)$ from the other events. Intuitively this should be possible because when $D(n)$ occurs, after the invasion touches $\mathcal{C}_*$, it does not need to check any weights for edges which are p_c-closed until after time T_{D_*}. To formally prove this, we represent the weights $(\omega(e))$ used for the invasion as functions of three independent variables. This representation is used in the "percolation cluster method" of Chayes-Chayes-Newman, but their method uses them in a dynamic way, whereas ours will be static. For this representation, we assign different variables to the edges: let $(U_e^1, U_e^2, \eta_e)_{e \in \mathcal{E}^2}$ be an i.i.d. family of independent variables, where U_e^1 is uniform on $[0, p_c]$, U_e^2 is uniform on $(p_c, 1]$, and η_e is Bernoulli with parameter p_c. Then we set

$$\omega(e) = \begin{cases} U_e^1 & \text{if } \eta_e = 1 \\ U_e^2 & \text{if } \eta_e = 0. \end{cases}$$

Next, we define another invasion percolation process $(\hat{G}_n)$ (a sequence of growing subgraphs) as follows. If $D(n)$ does not occur, then $\hat{G}_n$ is equal to $(0, \{\})$ for all n (it stays at the origin with no edges). If $D(n)$ does occur, then $\hat{G}_n$ proceeds according to the usual invasion rules (with the weights $(\omega(e))$) until it reaches $\mathcal{C}_*$. After it contains a vertex of $\mathcal{C}_*$, it no longer checks the ω-value of any edge $\hat{e}$ with $\eta_{\hat{e}} = 0$ (it only checks the η-value). When there are no more edges with η-value equal to one for the invasion to invade, it stops (we set $\hat{G}_n$ to be constant after this time). Associated to this new invasion will be stopping times similar to t_{D_*} and T_{D_*}:

$$\begin{aligned} \hat{t}_{D_*} &= \text{ first time at which the new invasion invades an edge from } D_* \\ \hat{T}_{D_*} &= \text{ first time at which the new invasion invades the entire } p_c\text{-open cluster of } D_*. \end{aligned}$$

Note that if $D(n)$ does not occur, $\hat{t}_{D_*} = \hat{T}_{D_*} = \infty$, and that if $D(n)$ occurs, $\hat{T}_{D_*}$ equals the first time after which the graphs $\hat{G}_n$ become constant.

Given these definitions, the top equation of (3.5.4) equals

$$\mathbb{P}(U_e^2 \in (p_c, p_c + \epsilon], \eta_e = 0, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D(n), t_{D_*} \leq t_n < T_{D_*}).$$

We then claim that

$$\begin{aligned} &\mathbb{P}(U_e^2 \in (p_c, p_c + \epsilon], \eta_e = 0, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D(n), t_{D_*} \leq t_n < T_{D_*}) \\ = \; &\mathbb{P}(U_e^2 \in (p_c, p_c + \epsilon], \eta_e = 0, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D(n), \hat{t}_{D_*} \leq t_n < \hat{T}_{D_*}). \end{aligned} \tag{3.5.5}$$

This equation holds because when $D(n)$ occurs, $t_{D_*} = \hat{t}_{D_*}$ and $T_{D_*} = \hat{T}_{D_*}$. Indeed, if $D(n)$ occurs, then both invasions (G_n) and $(\hat{G}_n)$ are equal until they touch $\mathcal{C}_*$. After this time, the original invasion (G_n) does not invade any p_c-closed edges until time T_{D_*}, and neither does $(\hat{G}_n)$ (by definition). This shows (3.5.5).

Now that we have (3.5.5), we simply note that because $(\hat{G}_n)$ does not use any edges in $B(2n)^c$ that are p_c-closed, the the times $\hat{t}_{D_*}$ and $\hat{T}_{D_*}$ are independent of $(U_e^2)_{e\in B(2n)^c}$. Furthermore, the events $\{\eta_e = 0\}$, $\{e \overset{q_n}{\longleftrightarrow} \partial B(n)$ in $B(4n)\}$, and $D(n)$ are independent of $(U_e^2)_{e\in B(2n)^c}$, and $U_e^2 \in (p_c, p_c + \epsilon]$ depends only on $(U_e^2)_{e\in B(2n)^c}$. By independence, therefore, the lower equation of (3.5.5) is equal to

$$\frac{\epsilon}{1-p_c}\mathbb{P}(\eta_e = 0, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D(n), \hat{t}_{D_*} \le t_n < \hat{T}_{D_*}),$$

which equals the bottom equation in (3.5.4). This shows (3.5.4). □

Combining Lemma 3.5.3 with (3.5.3), and then reducing to the subevent $D^e(n)$ (recall this is the subevent of $D(n)$ on which the paths involved in $D(n)$ do not use the given $e \subset \text{Ann}(2n, 4n)$), we obtain

$$\begin{aligned}
&\mathbb{E}\Xi_{t_n} \\
\ge\; &\frac{\epsilon}{1-p_c}\mathbb{E}Y_n\mathbf{1}_{D(n)\cap\{t_{D_*}\le t_n < T_{D_*}\}} \\
\ge\; &\frac{\epsilon}{1-p_c}\sum_{e\subset \text{Ann}(2n,4n)} \mathbb{P}(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e(n), t_{D_*} \le t_n < T_{D_*}).
\end{aligned} \tag{3.5.6}$$

The most difficult part of the above sum is the term $t_n < T_{D_*}$. To ensure that this occurs, we will construct a large set of vertices in the exterior of D_* which will connect to D_* by p_c-open paths. To do this, we will need to use independence to separate the interior of D_* from its exterior, using the following two events, which comprise pieces of the event $D(n)$.

Definition 3.5.4. *For any circuit $\hat{D}_* \subset$ Ann$(8n, 16n)$ around the origin, define the event $D^e_{int}(n, \hat{D}_*)$ that the following hold.*

1. *There exists a q_n-open circuit around the origin in Ann$(n, 2n)$.*

2. *There exists an edge $f \subset$ Ann$(6n, 7n)$ with $\omega(f) \in (q_n, p_c)$ such that:*

   (a) *there exists a p_c-closed dual path P around the origin in Ann$(4n, 8n)^* \backslash \{f^*\}$ that is connected to the endpoints of f^* so that $P \cup \{f^*\}$ is a circuit around the origin, and*

   (b) *there exists a p_c-open path connecting an endpoint of f to $B(n)$ (avoiding e), and another disjoint p_c-open path connecting the other endpoint of f to $\hat{D}_*$.*

We also define the event $D_{ext}(n, \hat{D}_*)$ that the following hold.

1. *There exists a p_c-open path from $\hat{D}_*$ to $\partial B(16n)$.*

2. *$\hat{D}_*$ is the outermost p_c-open circuit in Ann$(8n, 16n)$.*

Directly from the definitions, we note that for any circuit $\hat{D}_* \subset$ Ann$(8n, 16n)$, $D^e_{int}(n, \hat{D}_*) \cap D_{ext}(n, \hat{D}_*)$ implies $D^e(n)$ (actually the union over $\hat{D}_*$ of this intersection is equal to $D^e(n)$), and the events $D^e_{int}(n, \hat{D}_*)$ and $D_{ext}(n, \hat{D}_*)$ are independent. Last, for distinct $\hat{D}_*$, the events $(D^e_{int}(n, \hat{D}_*) \cap D_{ext}(n, \hat{D}_*))_{\hat{D}_*}$ are disjoint. Decomposing (3.5.7) over the choice of the outermost circuit $\hat{D}_*$, we obtain that $\mathbb{E}\Xi_{t_n}(\epsilon)$ equals

$$\frac{\epsilon}{1-p_c} \sum_{e \subset \mathrm{Ann}(2n,4n)} \sum_{\hat{D}_*} \mathbb{P}\left(\begin{array}{c} \omega(e) > p_c, e \xleftrightarrow{q_n} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*), \\ D_{ext}(n, \hat{D}_*), t_{\hat{D}_*} \leq t_n < T_{\hat{D}_*} \end{array} \right).$$

(Here $t_{\hat{D}_*}$ and $T_{\hat{D}_*}$ are similar to t_{D_*} and T_{D_*} but defined for the detministic circuit $\hat{D}_*$.) Note that $\{t_{\hat{D}_*} \leq t_n\}$ depends only on the weights in the interior of $\hat{D}_*$, but $\{t_n < T_{\hat{D}_*}\}$ does not depend only on the exterior. To force this dependence, we simply create a large

p_c-open cluster in the exterior of $\hat{D}_*$. For our deterministic $\hat{D}_*$, let

$$Z(\hat{D}_*) = \#\{e \subset B(16)^c : \omega(e) < p_c,\ e \overset{p_c}{\longleftrightarrow} \hat{D}_*\}.$$

If $Z(\hat{D}_*) > Cn^2\pi(n)$ on $D^e_{int}(n, \hat{D}_*) \cap D_{ext}(n, \hat{D}_*)$, then $t_n < T_{\hat{D}_*}$. Since this event depends on variables for edges in the exterior of $\hat{D}_*$, we can use independence for the lower bound for $\mathbb{E}\Xi_{t_n}(\epsilon)$ of

$$\frac{\epsilon}{1-p_c} \sum_{e \subset \text{Ann}(2n,4n)} \sum_{\hat{D}_*} \Big[\mathbb{P}\left(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*), t_{\hat{D}_*} \leq t_n\right) \\ \times \mathbb{P}\left(D_{ext}(n, \hat{D}_*), Z(\hat{D}_*) > Cn^2\pi(n)\right)\Big]. \tag{3.5.7}$$

Note that only the first factor inside the double sum depends on e. To bound it, we give the next lemma.

Lemma 3.5.5. *There exists $\mathcal{C}_6$ and $C_{18} > 0$ such that for all $n \geq 1$, all $\hat{D}_*$ around the origin in Ann$(8n, 16n)$, and all $C \geq \mathcal{C}_6$,*

$$\sum_{e \subset Ann(2n,4n)} \mathbb{P}\left(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \textit{ in } B(4n), D^e_{int}(n, \hat{D}_*), t_{\hat{D}_*} \leq t_n\right) \geq c_6 n^2 \pi(n).$$

Proof. First note that for any $\hat{D}_*$, we have $t_{\hat{D}_*} \leq t_n$ whenever $R_{t_n} \geq 16n$. Therefore it will suffice to show a lower bound for

$$\sum_{e \subset \text{Ann}(2n,4n)} \mathbb{P}\left(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*), R_{t_n} \geq 16n\right).$$

To do this, we will show both a lower bound

$$\sum_{e \subset \text{Ann}(2n,4n)} \mathbb{P}\left(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\right) \geq c_7 n^2 \pi(n) \tag{3.5.8}$$

and an upper bound

$$\sum_{e \subset \mathrm{Ann}(2n,4n)} \mathbb{P}\left(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*), R_{t_n} < 16n\right) \leq \frac{c_7}{2} n^2 \pi(n), \tag{3.5.9}$$

for all n, so long as C is larger than some $\mathcal{C}_6$.

Inequality (3.5.9) is easier, so we start with it. First sum over e and then apply the Cauchy-Schwarz inequality to get the upper bound

$$\begin{aligned}
&\left(\mathbb{E}\left(\#\{e \subset \mathrm{Ann}(2n,4n) : e \overset{p_c}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\}\right)^2\right)^{1/2} (\mathbb{P}(R_{t_n} < 16n))^{1/2} \\
\leq &\left(\sum_{e,f \subset \mathrm{Ann}(2n,4n)} \mathbb{P}(e \overset{p_c}{\longleftrightarrow} \partial B(e, n/2), f \overset{p_c}{\longleftrightarrow} \partial B(f, n/2))\right)^{1/2} (\mathbb{P}(R_{t_n} < 16n))^{1/2}.
\end{aligned}$$

Here, for example, $B(f, n/2)$ is the box of sidelength n centered at the bottom-left endpoint of e. The fact that the sum is bounded by $(C_{13} n^2 \pi(n))^2$ follows from standard arguments, like those in [29, p. 388-391]. (See the upper bound for $\mathbb{E} Z_n(\ell_0)^2$ we give in full detail below (3.5.20) for a nearly identical calculation.) This gives us the bound

$$\text{LHS of (3.5.9)} \leq C_{13} n^2 \pi(n) \sqrt{\mathbb{P}(R_{t_n} < 16n)}.$$

Due to Lemma 3.2.1, given any c_7 from (3.5.8) (assuming we show that inequality, which we will in a moment), we can find $\mathcal{C}_6$ such that for $C \geq \mathcal{C}_6$,

$$C_{13} \sqrt{\mathbb{P}(R_{t_n} < 16n)} \leq c_7/2,$$

and this completes the proof of (3.5.9).

Turning to the lower bound (3.5.8), since $\omega(e)$ is independent of both events $\{e \overset{q_n}{\longleftrightarrow}$

$\partial B(n)$ in $B(4n)\}$ and $D^e_{int}(n, \hat{D}_*)$,

$$\begin{aligned}
&\sum_{e \subset \text{Ann}(2n,4n)} \mathbb{P}\left(\omega(e) > p_c, e \xleftrightarrow{q_n} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\right) \\
&= (1-p_c) \sum_{e \subset \text{Ann}(2n,4n)} \mathbb{P}\left(e \xleftrightarrow{q_n} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\right). \qquad (3.5.10)
\end{aligned}$$

Estimating each summand from below uses some standard gluing constructions (see [27, Thm. 1] or [30, Lemma 6.3] for some examples), so we will only indicate the main idea. It will suffice to lower bound the sum over only $e \subset \hat{B}_n := [-4n, -2n] \times [-2n, 2n]$. To construct the event $D^e_{int}(n)$, we build the event $\bar{D}(n)$, defined by the following conditions:

[a] There exists a q_n-open circuit around the origin in Ann$(n, 2n)$.

There exists an edge $f \subset B'(n)$:=Ann$(6n, 7n) \cap [6n, \infty)^2$ with $\omega(f) \in (q_n, p_c)$ such that:

[b] there exists a p_c-closed dual path P around the origin in Ann$(4n, 8n)^* \backslash \{f^*\}$ that is connected to the endpoints of f^* so that $P \cup \{f^*\}$ is a circuit around the origin, and

[c] there exists a p_c-open path connecting one endpoint of f to $B(n)$ and remaining in $[-n, \infty) \times \mathbb{R}$. Also, there exists another disjoint p_c-open path connecting the other endpoint of f to $\partial B(16n)$.

Figure 3.4 illustrates the event $\bar{D}(n)$.

The event described in $[b]$ guarantees item 2(a) in the definition of $D^e_{int}(n, \hat{D}_*)$. Since the event described in $[c]$ has a p_c-open path from $\partial B(n)$ to $\partial B(16n)$ containing f without using e, the event $[c]$ implies item 2(b) in the definition of $D^e_{int}(n, \hat{D}_*)$. Therefore, for any

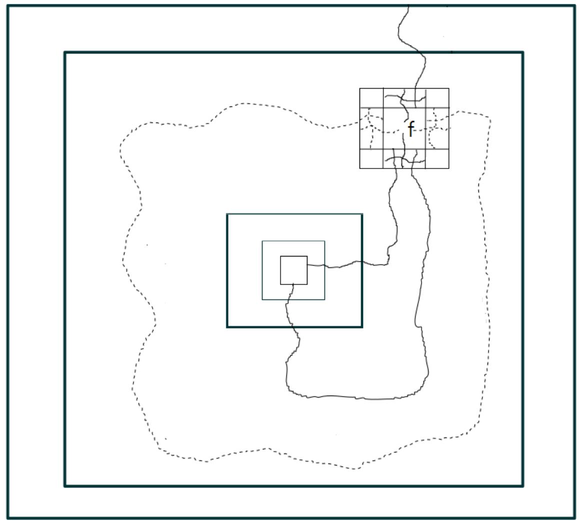

Figure 3.4: The event $\bar{D}(n)$. The boxes, in order from smallest to largest, are $B(n)$, $B(2n)$, $B(4n)$, $B(8n)$ and $B(16n)$. The solid circuit in Ann$(2n, 4n)$ is q_n-open. The solid paths from $\partial B(n)$ to f and the solid path from f to $\partial B(16n)$ are p_c-open. The dotted circuit in Ann$(4n, 8n)$ is p_c-closed.

circuit $\hat{D}^e_* \subset$ Ann$(8n, 16n)$, we can estimate the sum in the bottom of (3.5.10):

$$\sum_{e \subset \text{Ann}(2n,4n)} \mathbb{P}\left(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\right)$$
$$\geq \sum_{e \subset \hat{B}_n} \mathbb{P}\left(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \cap (-\infty, -n] \times \mathbb{R}, \bar{D}(n)\right). \qquad (3.5.11)$$

By applying the generalized FKG inequality (positive correlation for certain increasing and decreasing events, so long as they depend on particular regions of space — see [11, Lem. 13]) and a gluing construction, one can decouple the events described in $\bar{D}(n)$ and the event $\{e \overset{p_c}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \cap (-\infty, -n] \times \mathbb{R}\}$ to obtain the lower bound for (3.5.11) of

$$\sum_{e \subset \hat{B}_n} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \cap (-\infty, -n] \times \mathbb{R})\mathbb{P}([a])\mathbb{P}([b], [c])$$
$$\geq c_8 \mathbb{P}([b], [c]) \sum_{e \subset \hat{B}_n} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \cap (-\infty, -n] \times \mathbb{R}). \qquad (3.5.12)$$

To give a lower bound for $\mathbb{P}([b],[c])$, let $A(n,f)$ be the event described in [b] and [c] (along with the condition $\omega(f) \in (q_n, p_c)$), so that this probability equals $\mathbb{P}(\cup_f A(n,f))$, and the union is over $f \subset \mathrm{Ann}(6n,7n) \cap [6n,\infty)^2$. Letting $A'(n,f)$ be the same event, but with the p_c-open paths from [c] replaced by q_n-open paths, we obtain

$$\mathbb{P}([b],[c]) = \mathbb{P}(\cup_f A(n,f)) \geq \mathbb{P}(\cup_f A'(n,f)).$$

Note that the events $A'(n,f)$ for distinct f are disjoint. Therefore

$$\mathbb{P}([b],[c]) \geq \sum_f \mathbb{P}(A'(n,f)). \tag{3.5.13}$$

By a gluing argument involving the RSW theorem, the generalized FKG inequality, and Kesten's arms direction method (see [27, Eq. (2.9)]), if we define $B(n,f)$ as the event that there are two disjoint q_n-open paths connecting f to $\partial B(f,n)$, and two disjoint p_c-closed dual paths connecting f^* to $\partial B(f,n)$, then by using independence of the value of $\omega(f)$ from the event $A'(n,f)$, we can obtain

$$\mathbb{P}(A'(n,f)) \geq c_9(p_c - q_n)\mathbb{P}(B(f,n)). \tag{3.5.14}$$

Last, by a variant of [30, Lemma 6.3] (instead of taking $p,q \in [p_c,p_n]$, one takes $p,q \in [q_n,p_c]$, with $p = q_n$ and $q = p_c$, and the proof is nearly identical), we have $\mathbb{P}(B(f,n)) \asymp \mathbb{P}(A_n^{2,2})$, where $A_n^{2,2}$ is the four-arm event from (3.2.6). Using this with (3.5.13) and (3.5.14) gives

$$\mathbb{P}([b],[c]) \geq c_{10}(p_c - q_n)\sum_f \mathbb{P}(A_n^{2,2}).$$

By (3.2.6), we establish $\mathbb{P}([b],[c]) \geq c_{11}$, and putting this in (3.5.12),

$$\sum_{e \subset \hat{B}_n} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \cap (-\infty,-n] \times \mathbb{R}) \mathbb{P}([a]) \mathbb{P}([b],[c])$$
$$\geq c_8 c_{11} \sum_{e \subset \hat{B}_n} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \cap (-\infty,-n] \times \mathbb{R}). \tag{3.5.15}$$

Last, to deal with the summand of (3.5.15), we can use a gluing construction along with the FKG inequality and the RSW theorem to obtain

$$\mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n)) \geq c_{12} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(e, dist(e, \partial B(n))),$$

where $dist$ is the ℓ_∞-distance. By (3.2.1) and (3.2.5), we have

$$\mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(e, dist(e, \partial B(n)))) \geq c_{13} \mathbb{P}(e \overset{p_c}{\longleftrightarrow} \partial B(e, dist(e, \partial B(n)))) \geq c_{14} \pi(n).$$

Placing this in (3.5.15) and summing over e finally gives

$$\sum_{e \subset \text{Ann}(2n,4n)} \mathbb{P}\left(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\right) \geq c_{15} n^2 \pi(n),$$

which finishes the proof of (3.5.8).

□

Applying the lemma to the lower bound from (3.5.7), we obtain for all $C \geq \mathcal{C}_6$

$$\begin{aligned}
\mathbb{E}\Xi_{t_n}(\epsilon) &\geq \frac{\epsilon}{1-p_c} c_6 n^2 \pi(n) \sum_{\hat{D}_*} \mathbb{P}(D^e_{ext}(n, \hat{D}_*), Z(\hat{D}_*) > Cn^2\pi(n)) \\
&= \frac{\epsilon}{1-p_c} c_6 n^2 \pi(n) \mathbb{P}\left(\bigcup_{\hat{D}_*} \{D^e_{ext}(n, \hat{D}_*), Z(\hat{D}_*) > Cn^2\pi(n)\}\right) \\
&\geq \frac{\epsilon}{1-p_c} c_6 n^2 \pi(n) \mathbb{P}(A_n, B_n(C)),
\end{aligned}$$

where A_n is the event that there is a p_c-open circuit around the origin in $\text{Ann}(8n, 16n)$ and $B_n(C)$ is the event that there are more than $Cn^2\pi(n)$ vertices in $B(16n)^c$ connected to $B(16n)$ by p_c-open paths. By the FKG inequality and the RSW theorem,

$$\mathbb{E}\Xi_{t_n}(\epsilon) \geq \frac{\epsilon}{1-p_c} c_6 c_{16} n^2 \pi(n) \mathbb{P}(B_n(C)) \text{ for } n \geq 1, \text{ all } \epsilon > 0, \text{ and } C \geq \mathcal{C}_6. \tag{3.5.16}$$

Last, we argue that there exists a function F on $[0,\infty)$ such that $\inf_{r\in[0,m]} F(r) > 0$ for each $m \geq 0$ and such that

$$\mathbb{P}(B_n(C)) \geq F(C) \text{ for all } n \geq 1 \text{ and } C \geq 0. \tag{3.5.17}$$

Combining this with (3.5.16) and setting $G(C) = c_6 c_{16} F(C)/(1-p_c)$ will complete the proof of Proposition 3.5.1 and therefore of the proof of the upper bound in Theorem 3.3.1.

To show (3.5.17), we use some standard percolation arguments. For $\ell \geq 5$, set

$$Z_n(\ell) := \#\{v \in \text{Ann}(2^\ell n, 2^{\ell+1} n) : v \xleftrightarrow{p_c} \partial B(16n)\}.$$

By definition of $Z_n(\ell)$ and $B_n(C)$,

$$\mathbb{P}(B_n(C)) \geq \mathbb{P}(Z_n(\ell) > Cn^2\pi(n)) \text{ for any } \ell \geq 5. \tag{3.5.18}$$

To give a lower bound for the probability of $Z_n(\ell)$, we use the second moment method (Paley-Zygmund inequality):

$$\mathbb{P}\left(Z_n(\ell) \geq \frac{1}{2}\mathbb{E}Z_n(\ell)\right) \geq \frac{1}{4}\frac{(\mathbb{E}Z_n(\ell))^2}{\mathbb{E}Z_n(\ell)^2}. \tag{3.5.19}$$

Accordingly, we need a lower bound for $\mathbb{E}Z_n(\ell)$ and an upper bound for $\mathbb{E}Z_n(\ell)^2$.

To bound $\mathbb{E}Z_n(\ell)$ from below, note that if there is a p_c-open circuit around the origin in $\text{Ann}(2^{\ell+1}n, 2^{\ell+2}n)$ and a p_c-open path connecting $B(16n)$ to $\partial B(2^{\ell+2}n)$, then any point

$v \in \text{Ann}(2^\ell n, 2^{\ell+1}n)$ that is connected by a p_c-open path to $\partial B(v, 2^{\ell+3})$ (the box of side-length $2 \cdot 2^{\ell+3}$ centered at v) contributes to $Z_n(\ell)$. By the FKG inequality and the RSW theorem, then,

$$\mathbb{E}Z_n(\ell) \geq c_{17} f(\ell)\pi(2^{\ell+3}n)\#\{v : v \in \text{Ann}(2^\ell n, 2^{\ell+1}n)\}.$$

Here, c_9 is a lower bound for the probability of existence of the circuit, $f(\ell) > 0$ is a lower bound (depending only on ℓ) for the probability of a connection between the two boxes, and $\pi(2^{\ell+3}n)$ is the probability corresponding to the connection between v and $\partial B(v, 2^{\ell+3}n)$. By (3.2.5), we obtain

$$\mathbb{E}Z_n(\ell) \geq \left[c_{17}\frac{D_1}{\sqrt{2^{\ell+3}}}2^{2\ell}\right] n^2\pi(n).$$

If we fix $\ell = \ell_0$ so large that this is bigger than $2Cn^2\pi(n)$ for all n, we obtain from (3.5.18) and (3.5.19) that

$$\mathbb{P}(B_n(C)) \geq \frac{C^2(n^2\pi(n))^2}{\mathbb{E}Z_n(\ell_0)^2}. \tag{3.5.20}$$

For the upper bound on $\mathbb{E}Z_n(\ell_0)^2$, we follow the strategy of Kesten in [29, p. 388-391]. First note that any v counted in $Z_n(\ell_0)$ must have a p_c-open path connecting it to $\partial B(v, 2^{\ell_0-1}n)$. Therefore by independence,

$$\begin{aligned}
\mathbb{E}Z_n(\ell_0)^2 &\leq \sum_{v,w \in \text{Ann}(2^{\ell_0}n, 2^{\ell_0+1}n)} \mathbb{P}(v \overset{p_c}{\longleftrightarrow} \partial B(v, 2^{\ell_0-1}n), w \overset{p_c}{\longleftrightarrow} \partial B(w, 2^{\ell_0-1}n)) \\
&\leq \sum_{v \in \text{Ann}(2^{\ell_0}n, 2^{\ell_0+1}n)} \sum_{k=0}^{2^{\ell_0+2}n} \sum_{w : \|v-w\|_\infty = k} \pi(k/2)\pi(k/2)\pi(2k, 2^{\ell_0-1}n). \qquad (3.5.21)
\end{aligned}$$

Here, $\pi(2k, 2^{\ell_0-1}n)$ is the probability that there is an open path connecting $B(2k)$ to $\partial B(2^{\ell_0-1}n)$. (If $2k \geq 2^{\ell_0-1}n$, this probability is one.) By quasimultiplicativity [11, Eq. (4.17)] and the RSW theorem, we have

$$\pi(k/2)\pi(2k, 2^{\ell_0}n) \leq C_{14}\pi(2^{\ell_0}n),$$

which is itself bounded by $C_{14}\pi(n)$, so putting this in (3.5.21), we have an upper bound

$$\mathbb{E}Z_n(\ell_0)^2 \leq \left[C_{14}2^{2\ell_0}\sum_{k=0}^{2^{\ell_0+2}n}\pi(k)\right]n^2\pi(n).$$

By [29, Eq. (7)], we have $\sum_{k=0}^{2^{\ell_0+2}n}\pi(k) \leq C_{15}2^{2(\ell_0+2)}n^2\pi(n)$, and so we finish with $\mathbb{E}Z_n(\ell_0)^2 \leq C_{16}(n^2\pi(n))^2$, where C_{16} depends only on ℓ_0. Putting this into (3.5.20) finishes the proof of (3.5.17). $\square$

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